**Algebra of Time Series**
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Time series analysis relies heavily on the lag operator and other operators derived from the lag operator. The computation with these operators is isomorphic to a polynomial. Therefore an understanding of polynomial algebra is crucially important in time series analysis.

## Polynomial Equations

A general polynomial equation of complex variable, $z$, can be written as below:

$$g(z) = \phi_0 + \phi_1 z + \phi_2 z^2 + \cdots + \phi_n z^n = 0$$

It can be factored with the roots of the equation, $\lambda_i$:

$$\phi_n (z - \lambda_1)(z - \lambda_2) \ldots (z - \lambda_n) = 0$$

This expression can further be expanded as below:

$$\phi_n z^n - \phi_n \sum_{i=1}^{n} \lambda_i z^{n-1} + \phi_n \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j z^{n-2} - \cdots + (-1)^n \phi_n \prod_{i=1}^{n} \lambda_i = 0$$

Comparing the constant term in above expression with that of the first equation, the constant is:

$$\phi_n = \phi_0 \prod_{i=1}^{n} \frac{1}{(-\lambda_i)}$$

Put all these together, the general expression of a polynomial equation in terms of it roots becomes:

$$\begin{aligned} 0 = \sum_{i=0}^{n} \phi_i z^i &= \phi_n \prod_{i=1}^{n} (z - \lambda_i) \\ &= \phi_0 \prod_{i=1}^{n} \frac{1}{(-\lambda_i)} \prod_{i=1}^{n} (z - \lambda_i) \\ &= \phi_0 \prod_{i=1}^{n} \left(1 - \frac{z}{\lambda_i}\right) \end{aligned}$$

## Rational Functions of Polynmials

If $f(z)/g(z) = f(z)/\left[g_1(z) g_2(z)\right]$ is a rational function, and $g_1(z), g_2(z)$ have no common factor, then the function can be uniquely expressed as a sum of two rational functions:

$$\frac{f(z)}{g(z)} = \frac{f_1(z)}{g_1(z)} + \frac{f_2(z)}{g_2(z)}$$

Let $g(z) = \prod(1 - z/\lambda_i)$, repeatedly applying the above theorem yields the following:

$$\frac{f(z)}{g(z)} = \frac{c_1}{1 - z/\lambda_1} + \frac{c_2}{1 - z/\lambda_2} + \cdots + \frac{c_n}{1 - z/\lambda_n}$$

where $c_1, c_2, \ldots, c_n$ are constants and can be solved by adding the terms on the right hand side and equaling the numerator to $f(z)$.

<u>Lag Operator</u>

A time series is a sequence $x(t) = \{x_t; t = 0, \pm 1, \pm 2, \ldots\}$ where $t$ takes only integers. A lag operator, $L$, moves time one step backwards and is defined as below:

$$Lx(t) = x(t-1)$$

The lag operator can be applied consecutively; and generally,

$$L^k x(t) = x(t-k)$$

Further, we can define polynomials of the lag operator as:

$$p(L) = \prod_{i=0}^{n} p_i L^i = p_0 + p_1 L + \cdots + p_n L^n$$

and the effects on the time series sequence are:

$$p(L)x(t) = p_0 x(t) + p_1 x(t-1) + \cdots + p_n x(t-n) = \sum_{i=0}^{n} p_i x(t-i)$$

<u>Linear Difference Equations</u>

Given an n-th order linear difference equation:

$$\alpha_0 x(t) + \alpha_1 x(t-1) + \cdots + \alpha_n x(t-n) = u(t)$$

where $u(t)$ is the forcing function, the equation can be expressed using the lag operator:

$$\alpha(L)x(t) = u(t)$$

where $\alpha(L) = \alpha_0 + \alpha_1 L + \cdots + \alpha_n L^n$.

The equation may be solved in three steps. First, a general solution of the homogeneous equation (where $u(t) = 0$) with several unknown constants is found. Second, the particular solution with a given forcing function, $u(t) \neq 0$, is identified. Lastly, the two solutions are combined and the initial conditions are used to find the unknown constants.

If $\lambda_i$ is a root of $\alpha(z) = 0$, then $x_i(t) = (1/\lambda_i)^t$ is a solution of the homogeneous difference equation. This can be shown as below.

$$
\begin{aligned}
\alpha(L)x(t) &= \alpha_0 \prod_{i=1}^{n} \left(1 - \frac{L}{\lambda_i}\right) x(t) \\
&= \alpha_0 \prod_{i=1}^{n} \left(1 - \frac{L}{\lambda_i}\right) \left(\frac{1}{\lambda_j}\right)^t \\
&= \alpha_0 \prod_{i\neq j}^{n} \left(1 - \frac{L}{\lambda_i}\right) \left(1 - \frac{L}{\lambda_j}\right) \left(\frac{1}{\lambda_j}\right)^t \\
&= \alpha_0 \prod_{i\neq j}^{n} \left(1 - \frac{L}{\lambda_i}\right) \left[\left(\frac{1}{\lambda_j}\right)^t - \left(\frac{1}{\lambda_j}\right)^t\right] = 0
\end{aligned}
$$

The general solution of the homogeneous equation becomes:

$$
x(t;c) = c_1 \left(\frac{1}{\lambda_1}\right)^t + c_2 \left(\frac{1}{\lambda_2}\right)^t + \cdots + c_n \left(\frac{1}{\lambda_n}\right)^t
$$

where $c_i$ are the constants for the linear combination. If there are $m$ coincide roots of $\lambda_j$ then all of $(1/\lambda_j)^t, t(1/\lambda_j)^t, t^2(1/\lambda_j)^t, \ldots, t^{m-1}(1/\lambda_j)^t$ are solutions to the homogeneous equation.

If the homogenous equation has complex roots, they come with pairs of complex conjugates: $\mu = 1/\lambda, \mu^* = 1/\lambda^*$. and $\|\mu\| = \mu\mu^* = \kappa^2$. Therefore:

$$
\begin{aligned}
x_1(t) &= c\mu^t + c^*(\mu^*)^t \\
&= \rho e^{-i\theta}(\kappa e^{i\omega})^t + \rho e^{i\theta}(\kappa e^{-i\omega})^t \\
&= \rho\kappa^t \left[e^{i(\omega t - \theta)} + e^{-i(\omega t - \theta)}\right] \\
&= 2\rho\kappa^t cos(\omega t - \theta)
\end{aligned}
$$

In general,

$$
x(t) = 2\sum_{j=1}^{n/2} \rho_j \kappa_j^t cos(\omega_j t + \theta_j)
$$

In order for $x(t)$ to be stable (bounded or stationary), $\|\kappa_j\|$ must be less than unity; therefore the complex roots of $\alpha(z) = 0$, $\lambda_i$, must be outside the unit circle.

<u>Transfer Functions</u>

Given a linear difference equation of response variable $y(t)$, explanatory variable $x(t)$, and a noise term without a lag operator (*i.e.,* uncorrelated white noise):

$$
\alpha(L)y(t) = \beta(L)x(t) + \epsilon(t)
$$

The transfer function is the explicit expression for the response variable $y(t)$:

$$y(t) = \frac{\beta(L)}{\alpha(L)} x(t) + \frac{1}{\alpha(L)} \epsilon(t)$$

where

$$\alpha(L) = 1 + \sum_{i=1}^{p} \alpha_i L^i = 1 - \sum_{i=1}^{p} \phi_i L^i$$

$$\beta(L) = \sum_{i=0}^{k} \beta_i L^i$$

and $\alpha_i = -\phi_i$. The rational function can be expanded as:

$$\frac{\beta(z)}{\alpha(z)} = \omega(z) = \left\{\omega_0 + \omega_1 z + \omega_2 z^2 + \dots\right\}$$

The sequence of the coefficients $\{\omega_0, \omega_1, \omega_2, \dots\}$ constitutes the impulse response of the transfer function. In other words, if the input signal, $x(t)$, is a sequence of impulse in the form $\{\dots, 0, 1, 0, 0, 0, \dots\}$, *i.e.*, all zero values except only one instant, then the response sequence is in the form $\{\dots, 0, \omega_0, \omega_1, \omega_2, \dots\}$. The step response, on the other hand, is the partial sum of the coefficients. Specifically, if the input signal, $x(t)$, is a sequence of zeros up to a point and becomes constant one in the form $\{\dots, 0, 1, 1, 1, \dots\}$, then the response sequence is in the form $\{\dots, 0, \omega_0, \omega_0 + \omega_1, \omega_0 + \omega_1 + \omega_2, \dots\}$. The procedure of finding the coefficients of the transfer function can be illustrated using a simple second order system:

$$\frac{\beta_0 + \beta_1 z}{1 - \phi_1 z - \phi_2 z^2} = \omega_0 + \omega_1 z + \omega_2 z^2 + \dots$$

$$\beta_0 + \beta_1 z = \left(1 - \phi_1 z - \phi_2 z^2\right)\left(\omega_0 + \omega_1 z + \omega_2 z^2 + \dots\right)$$

$$\begin{aligned}
\omega_0 &= \beta_0 \\
\omega_1 &= \beta_1 + \phi_1 \omega_0 \\
\omega_2 &= \phi_1 \omega_1 + \phi_2 \omega_0 \\
&\vdots \\
\omega_n &= \phi_1 \omega_{n-1} + \phi_2 \omega_{n-2}
\end{aligned}$$

From above equation, the necessary and sufficient condition for $\{\omega_i\}$ to converge is that the roots ($\lambda_i$) of autoregressive equation $1 + \alpha_1 z + \alpha_2 z^2 = 0$ [NOTE: $\alpha_i = -\phi_i$] to be outside the unit circle.

Given an input $x(t) = cos(\psi t)$ without noise, the output through the transfer function becomes:

$$
\begin{aligned}
y(t) &= \frac{\beta(L)}{\alpha(L)}x(t) = \omega(L)x(t) = \sum_{j=0}^{k} \omega_j cos(\psi\,[t-j]) \\
&= cos(\psi t)\sum_{j=0}^{k} \omega_j cos(\psi j) + sin(\psi t)\sum_{j=0}^{k} \omega_j sin(\psi j) \\
&= \alpha cos(\psi t) + \beta sin(\psi t) = \rho cos(\psi t - \theta)
\end{aligned}
$$

The trigonometrical equality $cos(\alpha-\beta) = cos(\alpha)cos(\beta) + sin(\alpha)sin(\beta)$ is used and:

$$
\alpha = \sum_{j=0}^{k} \omega_j cos(\psi j)
$$

$$
\beta = \sum_{j=0}^{k} \omega_j sin(\psi j)
$$

$$
\rho = \sqrt{\alpha^2 + \beta^2}
$$

$$
\theta = tan^{-1}\left(\frac{\beta}{\alpha}\right)
$$

Therefore, the effect of a transfer function on a signal, $x(t) = cos(\psi t)$, is not only altering its amplitude but adding a phase shift as well, $y(t) = \rho cos(\psi t - \theta)$.

## Concluding Remarks

The computation of the lag operator in time series is isomorphic to ordinary polynomials. Using the algebra of polynomials, it is shown that lag operator polynomials can be expressed in terms of their roots. The time series can further be expressed as a difference equation. The stability of the time series is determined by whether the roots of the autoregressive polynomial are outside the unit circle. Transfer functions are other ways to explicitly show the response of the system to a given input signal without the effect of a noise term. Again the stability of the system relies critically on the locations of the “poles” (*i.e.,* the roots of the autoregression or feedback of the system polynomial).