Principles of Quantum Mechanics
02/27/2026

Descriptions of microscopic world require quantum mechanics to include new set of matter interactions at such small scale. However, the ideas of quantum mechanics are beyond ordinary human intuition, therefore abstract mathematics (or more precisely abstract algebra) is used as a tool to convey these ideas completely without distortion. This article summarizes the five principles of quantum mechanics.

Vector Space

A quantum state is described by a vector in a linear vector space and the vector is denoted by a *ket*: $|A\rangle$. The ket operation is closed with properties of additivity, commutativity, and associativity:

$$|A\rangle + |B\rangle = |C\rangle$$

$$|A\rangle + |B\rangle = |B\rangle + |A\rangle$$

$$(|A\rangle + |B\rangle) + |C\rangle = |A\rangle + (|B\rangle + |C\rangle)$$

$|0\rangle$ or 0 is the addition identity element in the vector space. The commutativity and associativity of the operation also extend to the multiplication between a ket and a complex scalar, z.

The complex conjugation of the ket is called *bra*, denoted as $\langle A|$. The inner product of bra and ket is *bracket*, $\langle A|A\rangle = \|A\|^2$.

Orthonormal Basis

Let $|i\rangle$ be an orthonormal basis in an N-dimensional vector space, any ket can be represented as:

$$|A\rangle = \sum_i \alpha_i \, |i\rangle$$

where α_i, generally a complex number, is the component of the vector on the *i-th* basis vector. The component can be computed using the fact that the inner product of two different orthonormal basis vectors is zero.

$$\langle j|A\rangle = \sum_i \alpha_i \, \langle j|i\rangle = \alpha_j$$

Therefore, a ket vector can be written in a more elegant form:

$$|A\rangle = \sum_i |i\rangle \, \langle i|A\rangle$$

Linear Operators

A mapping or projection of a ket is done using matrix operation. The operation is also closed with corresponding additivity, commutativity, and associativity:

$\mathbf{M}\,|A\rangle = |B\rangle$

However, matrix multiplicative operations are not commutable. The multiplicative identity element is the identity matrix $\mathbf{I}$. Physically, matrix operations produce observables of the quantum states that one can measure. An example is Hamiltonian matrix operator on a quantum state yielding the energy level of the state. In general, a matrix operator on a ket will reorient the vector to a different ket. However, if the ket is one of the eigenvectors of the matrix operation, the orientation of the ket does not change. One may also normalize the eigenvectors of the matrix operator and form a complete orthonormal basis. A matrix operator can also apply to a bra. The component forms of the above matrix operation on $|A\rangle$ and its complex conjugate are:

$$\sum_i m_{ji}\alpha_i = \beta_j$$

$$\sum_i m_{ji}^*\alpha_i^* = \beta_j^*$$

where β_j is the component of $|B\rangle$ and $m_{ij} = \langle i|\mathbf{M}|j\rangle$. Let $\mathbf{M}^\dagger = (\mathbf{M}^T)^*$, the matrix operator on bra $\langle A|$ becomes:

$\langle A|\,\mathbf{M}^\dagger = \langle B|$

Let λ and $|\lambda\rangle$ be respectively the eigenvalue and eigenvector of matrix operator $\mathbf{M}$:

$\mathbf{M}|\lambda\rangle = \lambda|\lambda\rangle$

$\langle\lambda|\mathbf{M}^\dagger = \langle\lambda|\lambda^*$

Taking the inner products of above equations:

$\langle\lambda|\mathbf{M}|\lambda\rangle = \langle\lambda|\lambda\rangle\lambda$

$\langle\lambda|\mathbf{M}^\dagger|\lambda\rangle = \langle\lambda|\lambda\rangle\lambda^*$

For a physical observable, the eigenvalue λ must be a real number, i.e., λ equals its complex conjugate λ^*; as such, the matrix operator $\mathbf{M}$ must be the same as the complex conjugation of its transpose, $\mathbf{M}^\dagger$. Matrices with such a property are called *Hermitian matrices* after Charles Hermite, a French mathematician. The corresponding matrix operators are called Hermitian operators.

<u>Principle One</u>

The observable (measurable) quantities of quantum mechanics can be represented by matrix operators in a vector space. The vector space is closed with commutative and associative addition operations. The matrix group is closed through associative addition operations and multiplicative operations. The addition operations may be commutative, but the multiplicative operations are not. These matrix operators must be Hermitian operators, *i.e.*, the matrix is the same as the complex conjugate of its transpose.

Principle Two

The observables are eigenvalues of the Hermitian operator and the eigenvalues are real numbers. The quantum states as results of the (measurement) operator corresponding to the eigenvectors of the matrix operator. In other words, all possible measurement values are the eigenvalues of the matrix operator; other values cannot be the measurement outcome. For example, if the eigenvalues of a spin operator is either plus one or minus one, a zero cannot be a valid measurement, even though the statistical average of the measurements may be zero.

Principle Three

Distinct measurement states are eigenvectors the matrix operator. These eigenvectors form orthonormal basis of the vector space. For example, an upper spin state (with eigenvalue +1) is orthogonal to the down spin state (with eigenvalue -1), even though they may not be orthogonal in a geometrical sense.

Principle Four

The probability of observing a specific eigenvalue λ_i of a Hermitian operator for a given state $|A\rangle$ is:

$$P(\lambda_i) = \|\langle A|\lambda_i\rangle\|^2 = \langle A|\lambda_i\rangle\langle\lambda_i|A\rangle$$

The probability is the square of the projection of state $\langle A|$ on the eigenvector $|\lambda_i\rangle$ normalized to the probabilities of all other eigenvectors. As such, the average value of the observations of an operator as computed using the probability distribution of all eigenvectors could be analogous to the result of a similar operator in classical mechanics.

Principle Five

The evolution a quantum state $|\Psi(t)\rangle$ with time is represented by the time evolution operator $\mathbf{T}$ on the state at a previous time, e.g., at time zero $|\Psi(0)\rangle$:

$$|\Psi(t)\rangle = \mathbf{T}|\Psi(0)\rangle$$

Given a distinct different state $|\Phi(0)\rangle$, it must be orthogonal to $|\Psi(0)\rangle$ at time zero (see Principle Three), *i.e.*,

$$\langle\Phi(0)|\Psi(0)\rangle = 0$$

For a physical system that keeps quantum states separate (information conservation), this distinction between states must be preserved for any future time:

$$\langle\Phi(t)|\Psi(t)\rangle = 0$$

Therefore, this leads to the fact:

$$\mathbf{T}^\dagger\mathbf{T} = \mathbf{I}$$

The time evolution operator $\mathbf{T}$ is said to be *unitary*.