The Kelly Criterion

The Kelly Criterion was first introduced in 1956 (*"A New Interpretation of Information Rate"*) in solving the St. Petersburg Paradox (a coin flip game). It was later extended by Edward Thorp (*"Beat the Dealer"*) for card games. A recent writing by Thorp went through a great length of the treatment (*"The Kelly Criterion in Blackjack, Sport Betting, and the Stock Market"*) but a few key details were omitted. This article follows Jane Hung's treatment (*"Betting with the Kelly Criterion"*) with focus on using the binomial distribution (see Section 7.1 of Thorp's Thesis).

The mathematical technique to approximate the random variable X_i using a binomial distribution with two distinct values at $\mu + \sigma$ and $\mu - \sigma$ each with equal probability of $\frac{1}{2}$, *i.e.,*

$$P(X_i = \mu + \sigma) = P(X_i = \mu - \sigma) = \frac{1}{2} \tag{1}$$

This binomial distribution has a mean of μ and variance of σ^2.

Assuming that the initial asset is Y_0, risk-free annual interest rate r, and betting fraction (position sizing) f. At the n-th compounding cycle within a year, the total asset Y_n grows into,

$$Y_n = Y_0 \prod_{i=1}^{n} \left(1 + (1-f)\frac{r}{n} + fX_i\right) \tag{2}$$

The expected logarithmic growth of the asset is:

$$E\left[ln\left(\frac{Y_n}{Y_0}\right)\right] = E\left[\sum_{i=1}^{n} ln\left(1 + (1-f)\frac{r}{n} + fX_i\right)\right] \tag{3}$$

Accounting for the binomial distribution described in equation (1) for X_i

$$
\begin{aligned}
E\left[ln\left(\frac{Y_n}{Y_0}\right)\right] &= \sum_{i=1}^{n} \frac{1}{2} ln\left(1 + (1-f)\frac{r}{n} + f\left(\frac{\mu}{n} + \frac{\sigma}{\sqrt{n}}\right)\right) \\
&+ \frac{1}{2} ln\left(1 + (1-f)\frac{r}{n} + f\left(\frac{\mu}{n} - \frac{\sigma}{\sqrt{n}}\right)\right) \\
&= \frac{n}{2} ln\left(\left(1 + (1-f)\frac{r}{n} + \frac{f\mu}{n}\right)^2 - \left(\frac{f\sigma}{\sqrt{n}}\right)^2\right)
\end{aligned} \tag{4}
$$

Carrying out the Taylor expansion of the above equation at $f = 0$ and taking n to infinite (continuously compounding),

$$
E\left[ln\left(\frac{Y_\infty}{Y_0}\right)\right] = r + (\mu - r)f - \frac{(f\sigma)^2}{2} \tag{5}
$$

Solving the quadratic equation of f to maximize the logarithmic growth, the Kelly Criterion for the optimal betting fraction f^* is

$$
f^* = \frac{\mu - r}{\sigma^2} \tag{6}
$$