Ito's Lemma

Ito's Lemma is an identity to find the differentiation of a time-dependent *function* of a stochastic process. Given a stochastic differential equation:

$$dX_t = \mu_t dt + \sigma_t dw_t \tag{1}$$

where w_t is a Wiener process. However for a complex stochastic process, $f(t)$, a function of X_t, Ito's Lemma is equivalent to the chain rule in differentiation.

$$df(t) = a(f(t), t)dt + b(f(t), t)dw_t \tag{2}$$

For $f(t, X_t)$ as a differentiable function, its Taylor expansion is

$$df = \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}dt^2 + \frac{\partial f}{\partial X}dX + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}dX^2 + \cdots \tag{3}$$

Substituting $dX_t = \mu_t dt + \sigma_t dw_t$ and note that when dt approaches 0, the terms dt^2 and $dtdw_t$ go to zero faster than dw_t^2, and $dw_t = \sqrt{dt}$. The Ito's Lemma is

$$df = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma_t \frac{\partial f}{\partial x}dw_t \tag{4}$$

One may compare equation (4) with (2) and get the sense for a() and b(). For a geometric Brownian motion: $dS_t = \mu S_t dt + \sigma S_t dw_t$, and noting $f(S_t) = ln(S_t)$,

$$\begin{aligned}
dln(S_t) &= f'(S_t)dS_t + \frac{1}{2}f''(S_t)(dS_t)^2 && \text{Taylor expansion of } f \\
&= \frac{1}{S_t}dS_t - \frac{1}{2S_t^2}(dS_t)^2 && \text{carry out } f' \text{ and } f'' \\
&= \frac{1}{S_t}\left(\mu S_t dt + \sigma S_t dw_t\right) - \frac{1}{2}\sigma^2 dt && \text{substituting } dS_t \text{ and } dw_t^2 = dt \\
&= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dw_t
\end{aligned} \tag{5}$$