Determinants
04/05/2025

Determinant of a square matrix is a scalar value of the matrix and is introduced as part of computational procedure or as the criterion for the invertibility of the matrix. The full picture of this quantity and the applications are often scattered in the literature. This article is a short summary of determinants and their applications in linear algebra.

Cofactor of C_{ij} of Matrix A

The formal definition of determinants (e.g., $detA$) is recursive. First, we define the cofactors of a square matrix A. For a square matrix A, the minor of entry a_{ij}, denoted as $detA_{ij}$, is the determinant of the remaining submatrix after deleting the i-th row and j-th column from matrix A. $(-1)^{i+j}detA_{ij}$, denoted as C_{ij}, is the cofactor of matrix entry a_{ij}.

We see that the cofactor is the minor entry at the i-th row and j-th column with its sign alternating from one to the immediate next in the matrix.

Formal Definition

The determinant of an n x n matrix A is the sum of the products of the first row entries and its corresponding cofactors. This is also called the first row cofactor expansion.

$$detA = \sum_{j=1}^{n} a_{1j}C_{1j}$$

This is indeed a recursive definition since the cofactors are defined as the determinants of some submatrices of A with alternating signs.

The computation of determinants does not have to follow the first row expansion as defined above. Instead, the determinant can be computed with any row or any column cofactor expansion. The following is regarded as a determinant computational procedure, rather than a definition in most textbooks. i-th row expansion:

$$detA = \sum_{j=1}^{n} a_{ij}C_{ij}$$

j-th column expansion:

$$detA = \sum_{i=1}^{n} a_{ij}C_{ij}$$

The logic for the computation of determinants by any row or any column expansion will become clear after examining the matrix row operations.

<u>Row Operations</u>

An elementary matrix produces matrix operation by left multiplying the matrix. The elementary matrix (*E*) itself is generated by performing the same row operation on the identity matrix. For example,

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

exchanges the first row and the second row - row swapping. det E = -1.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

divides the second row by a factor of 5 - row scaling. det E = 1/5.

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

multiplies the first row by -2 and adds it to the second row - row replacement. det E = 1.

The row operation theorem of a square matrix A has the following propositions.

A) If matrix B is produced by row replacement of matrix A then det B = det A
B) If matrix B is produced by row scaling matrix A by a factor of k then det B = k det A
C) If matrix B is produced by one time row swapping matrix A then det B = - det A

Further, if matrix B is produced by transpose of matrix A then det B = det A^T = det A.

The proof of these is by mathematical induction on the size of the matrix. The effects on the determinant of matrix A by these row and transpose operations suggest that the cofactor expansion by any row or any column of a matrix is equivalent to the cofactor expansion by the first row - as the recursive definition of matrix determinant.

When successive row operations produce the matrix in an echelon form, successive first column cofactor expansion shows that the determinant of the matrix becomes the product of the matrix diagonal entries in an echelon form.

Invertibility of a Square Matrix

The invertibility theorem states that a square matrix A is invertible if and only if det A $\neq$ 0. If matrix A is invertible then equation A**x**=0 has only non-trivial solutions therefore all the diagonal entries of the matrix in an echelon form must not be zero. Conversely, if det A $\neq$ 0, the diagonal entries of the matrix in an echelon form are not zero, therefore equation A**x**=0 only has non-trivial solutions and matrix A is invertible.

Geometric Interpretation

The magnitude of the determinant of a square matrix ($|detA|$) is the "volume" formed by the columns of matrix A. This can be understood by row operations of replacement and swap only (no scaling) on matrix A to produce a diagonal matrix (not necessarily the row reduced echelon form). The product of the diagonal entries is the determinant. Also the columns form a subspace whose basis encloses a "volume" of the same value as the determinant. For a 2X2 matrix, this is the area of the parallelogram but for a 3X3 matrix, it is the volume of the parallelepiped.

Applications

Cramer's Rule: Let A be an invertible n x n matrix then for any **b** in R^n, the unique solution **x** of A**x**=**b** has entries of **x** given below:

$$x_i = \frac{detA_i(b)}{detA},\ i = 1, 2, \ldots, n$$

where $A_i(b)$ is the matrix constructed from A by replacing column i by the vector **b**.

However, Cramer's rule is quite inefficient in solving A**x**=**b** in practice in comparison with other methods. It has theoretical value for analyzing the sensitivity of a solution in response to changes in **b**.

A Formula for A^{-1}: Let A be an invertible n x n matrix, then

$$A^{-1} = \frac{1}{detA} adjA$$

where adj A is a matrix formed by the corresponding cofactors C_{ij} as its entries.

Characteristic Polynomial: The eigenvalues (λ) of an invertible square matrix, A, satisfies $(A - \lambda I)\mathbf{x} = \mathbf{0}$ and has a nontrivial solution. Therefore, the matrix $A - \lambda I$ must *not* be invertible. The condition becomes that the determinant of matrix $A - \lambda I$ is zero:

$$det(A - \lambda I) = 0$$

The expansion of the equation above is the characteristic polynomial and the solutions are the eigenvalues of matrix A.

Concluding Remarks

The definition of determinant is based on a computational procedure and does not give a concrete visual of the quantity. The geometric interpretation, however, reveals the volumetric nature of this quantity through matrix row operations that show the same multiple of the basis axes i.e., a "volume", and the diagonal product whose value is the determinant.

This makes a lot of sense. Any basis axis of the subspace spanned by the column vectors must have a nonzero length. If one of the axes is degenerated and its length becomes zero so is the volume. Such a matrix is no longer invertible. In addition, through matrix row operations without scaling, the diagonal terms of RREF are also the eigenvalues (λI_n). Therefore, an invertible matrix whose eigenvalues must not be zero.

The computation of determinant is best done using matrix row operations without scaling and the determinant is the diagonal product of the REF. The method of the original definition using cofactor expansion is cumbersome in comparison with row reduction.

The use of Camer's rule in solving linear equations has been replaced by computational approaches, such as LU-factorization. The practical use of this traditional method is very limited. The same is true for computation of A^{-1} using a determinant.

The above discussion shows a connection between eigenvalues of an invertible matrix and its determinant. They are the diagonal terms but one are those of the reduced row echelon form and the other are those of the row echelon form, both without scaling operations.