Mathematics of Counting
02/23/2025

Counting is the oldest human encounter with numbers (or objects in their numeral characteristics). Although the modern treatment of counting has been based on Set Theory and Propositional Logic, the techniques used in counting have a long history since antiquity. Many scientific, engineering, business, and financial achievements rely on accurate counting of fundamental particles, crystal symmetries, permutations of business strategies, and the complete outcomes of an event that has financial impact. This article offers a brief summary of the counting techniques.

Sum Rule (Addition Rule)

If the outcome of an experiment can either be one of m outcomes or one of n outcomes, where none of the outcomes from the set with m outcomes is the same as any of the outcomes in the set with n outcomes (i.e., they are disjoint), there are total $m+n$ possible outcomes of the experiment.

Example: How many positive integers not exceeding 100 are divisible by 11 or 13? There are $m = \lfloor 100/11 \rfloor$ integers that are divisible by 11 and $n = \lfloor 100/13 \rfloor$ integers that are divisible by 13. There are no integers not exceeding 100 that are divisible by both 11 and 13. Therefore, $m + n = 16$.

Inclusion-Exclusion Principle

When the sets are not disjoint from each other, the Sum Rule becomes the Inclusion-Exclusion Principle. They become two count tasks with overlapping elements.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example: How many positive integers not exceeding 1000 are divisible by 7 or 11? There are $|A| = \lfloor 1000/7 \rfloor$ integers divisible by 7, $|B| = \lfloor 1000/11 \rfloor$ integers divisible by 11, and $|A \cap B| = \lfloor 1000/77 \rfloor$ integers divisible by both 7 and 11. $|A \cup B| = 142 + 90 - 12 = 220$.

Product Rule

If an experiment has two parts, where the first part can result in one of m outcomes and the second part can result in one of n outcomes regardless of the outcome of the first part, then the total number of possible outcomes for the experiment is $m \cdot n$.

Example: Two 6-sided dice are rolled. How many possible outcomes of the roll are there? The first die can come up with 6 possible outcomes. The second die can come up with 6 possible outcomes. The total number of possible outcomes is $6 \times 6 = 36$.

Permutations with Repetitions

This is a special case of the Product Rule. The number of distinct m-element sequences where each of the terms has one of n possible values (called "m-permutations of an n-element set") is given by the expression: n^m.

Example: The number of possible 10-element strings of lower-case English letters of alphabet is equal to $26^{10} = 141167095653376$.

Permutations of Distinct Objects

A permutation of a set S is a sequence that contains every element of S exactly once. Those n objects ($n = |S|$) can be permuted in $n \cdot (n-1) \cdot (n-2) \cdot ... \cdot 2 \cdot 1 = n!$ ways, since repetition is not allowed and each subsequent choice becomes one element less.

Example: How many ways can the letters ABCDEFGH be arranged such that the string ABC is a consecutive substring? The number of permutations is the same as the number of permutations of the six objects: ABC, D, E, F, G, and H, $6! = 720$.

Division Rule I

If the mapping from set A to set B is $k - to - 1$, then $|A| = k \times |B|$.

Example: In how many ways can King Arthur seat n different knights at his round table? Two seatings are considered equivalent if one can be obtained from the other by rotation. This mapping is an $n - \#to\# - 1$ function, since all n cyclic shifts of the original sequence map to the same seating arrangement.

$$\frac{n!}{n} = (n-1)!$$

Division Rule II

If the permutation of a set S has a few elements s_k repeated. The ordering of these repeated elements does not change the configuration.

$$\frac{|S|!}{\prod_{k=1}^{n} s_k!}$$

Example: How many anagrams can we build from “MATHEMATICA”? There are 11 letters; but “A” is repeated three times, “T” twice, and “M” twice:

$$\frac{11!}{3!2!2!} = 1663200$$

<u>Combinations of Distinct Objects</u>

Combination is to find the ways of selecting a set of k objects from n distinct objects where order is not important (so as these $(n-k)$ objects that are not selected). This is an extension to the Division Rule since changing the order in the selected objects and unselected objects does not alter the arrangement. This can be written as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Example: How many ways are there to select 3 books from a set of 6 distinct books? This is a straightforward 6-choose-3 combination problem.

$$\binom{6}{3} = \frac{6!}{3!3!} = 20\ ways.$$

<u>Combinations with Repetition</u>

When repetition of elements is allowed, it refers to a selection of items from a set where you can choose the same item multiple times, meaning the order of selection doesn't matter and each element can be picked more than once. This may also be understood as returning the $r-1$ previously selected items back to the original set to account for “replacements”. The number of possible ways that one selects r items from a set of n different items is:

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

Example: A mathematician is throwing a party, and she wants to set out 15 assorted cans of soft drinks for her guests. She shops at a store that sells five different types of soft drinks. How many different selections of cans of 15 soft drinks can she make?

$$\binom{5+15-1}{15} = \binom{19}{15} = 3876$$

Assignment of Indistinct Objects

Different from the combinations with repetition problem, the assignment problem is one where you want to place n indistinguishable items into r containers. The model is to place n items along with $r-1$ "separators" of the containers. Then the selection becomes to choose n items from the new $n+r-1$ mix.

$$\binom{n+r-1}{n}.$$

Example: You are a startup incubator and you have $10 million to invest in 4 companies (in $1 million increments). How many ways can you allocate this money?

$$\binom{10+4-1}{10} = \binom{13}{10} = 1716.$$

Concluding Remarks

The fundamental rules for counting are the sum (addition) rule and product rule. In cases where "over-counting" is made, the inclusion-exclusion principle is used to deduct (subtract) the extra items, and the division rules help to discount arrangements within subsets that do not change the configuration. Permutation can either allow replacements or prohibit replacement. Factorial is used for the latter situation. Again, division rules are employed when subset ordering does not change the configuration. Combinations are special cases for permutations where orders do not matter. However, combinations with repetition (replacement) require compensation to the original set by "returning" previously selected items. Further, the assignment problem requires an interesting model to compensate for the original set before the selections can be made by combination.