Mathematics of Interest Rate

An interest is paid to account for the time value of money when a lender provides money to fund the purchase of goods for the borrower. An interest rate is the ratio between the interest and the sum of money in the exchange. The relationship between interest rate and time is not linear thus different interest rates are required for different lengths of time of the borrowing. The time length is called the *term* of the loan. This article demonstrates various examples when interest rate is involved.

1 The Present Value

The future value of an asset (A_n) priced today as A_0 with a fixed (deterministic as opposed to stochastic) interest rate r is:

$$A_n = A_0 \cdot (1+r) \cdot (1+r) \cdots (1+r) = A_0 \dot{(}1+r)^n \tag{1}$$

where r is evaluated (compounded) for period (annually) 1, 2, ..., n, therefore called a *simple interest rate*. However, if the interest rate is compounded more frequently (e.g., monthly), the future value is then:

$$A_n = A_0 \cdot \left(1 + \frac{r}{m}\right)^{n \cdot m} \tag{2}$$

where m is the number of evaluations performed in each period (e.g., 12, or monthly). In a more theoretical sense, when the m approaches infinity, we have continous compounding:

$$A_n = A_0 \cdot \lim_{m \to \infty} \left(1 + \frac{r}{m}\right)^{n \cdot m} = A_0 \cdot e^{r \cdot n} \tag{3}$$

Please note, continuous compounding, or continuous interest rate, is not used in practical, day-to-day financial transactions. Conversely, if one is promised to receive a payment A_n after n years, the present value of such a payment (asset) is:

$$A_0 = \frac{A_n}{\left(1 + \frac{r}{m}\right)^{n \cdot m}} \tag{4}$$

The future value A_n is said to be discounted to the present value A_0 at constant interest rate r.

2 Pension and Annuity

A pension is a promise to provide a fixed monthly payment in the future throughout one's life time. To fund this plan, the pension provider must compute this promise in today's money (present value):

$$P_0 = p \cdot \sum_{i=1}^{\infty} \frac{1}{(1+r)^i} = \frac{p}{r} \lim_{n\to\infty} \left[1 - \frac{1}{(1+r)^n}\right] = \frac{p}{r} \tag{5}$$

where p is the annual pension payment. At an interest rate 4%, for example, the provider must secure \$1 million at the time of retirement to keep paying the retiree \$40,000 a year indefinitely. In reality, the retiree may only have 23 years to live given the life expectancy of a specific population (in the US, a typical early retirement age is 55 and the life expectancy is 78), this amount can be reduced to \$600,871, a significant saving to the provider.

NOTE: the sum of a geometric series is:

$$\sum_{i=1}^{n} a^i = a \cdot \left(1 + a + \cdots + a^{n-1}\right) = a \cdot \frac{1-a^n}{1-a} \tag{6}$$

For $a = 1/(1+r)$:

$$\sum_{i=1}^{n} \frac{1}{(1+r)^i} = \frac{1}{(1+r)} \cdot \frac{1-1/(1+r)^n}{1-1/(1-r)} = \frac{1}{r} \cdot \left(1 - \frac{1}{(1+r)^n}\right) \tag{7}$$

3 Consumer Loans

Similarly, a mortgage is a promise to pay a fixed monthly amount throughout the loan period, typically 30 years, to the loan provider (a bank). The bank needs to compute the discounted present value of this arrangement so that a loan in today's money can be issued as part of the house payment.

$$M_0 = \sum_{i=1}^{12\cdot yr} \frac{m}{\left(1+\frac{r}{12}\right)^i} = \frac{12m}{r} \cdot \left(1 - \frac{1}{(1+r/12)^{12yr}}\right) \tag{8}$$

where M_0 is the loan amount, m monthly payment, yr loan term (years), and r annual interest rate. To solve the required monthly payment, one may rearrange the above equation.

$$m = \frac{M_0 r}{12} \cdot \frac{(1+r/12)^{12yr}}{(1+r/12)^{12yr}-1} \tag{9}$$

The monthly payment for a 30-year home loan of \$300,000 at 7% interest rate is \$1996. The total payment through the term is in fact \$718560, more than twice of the loan value (140% paid for the finance charge). If the borrower decides to refinance it to a 15-year loan at an interest rate of 4%, the monthly payment increases to \$2219, but the total payment reduces to \$399,420; only 25% is the finance charge.

4 Funding a Pension Plan

Suppose that an employee is self funding a pension plan for the retirement. The monthly contribution (premium) must result in a sizable sum at the retirement age to sustain an annual pension payment through the rest of life. This monthly premium is:

$$m = p \cdot \frac{\sum_{i=1}^{78-R} \frac{1}{(1+r)^i}}{\sum_{j=1}^{12\cdot(R-25)} \left(1+\frac{r}{12}\right)^j} \tag{10}$$

where R is the retirement age, assuming that the employee started work at age 25 and the life expectancy is 78. At an interest rate 4%, the monthly premium is \$853.40 for a \$40K-a-year pension starting at age 55. If the employee postpones retirement until 65, the monthly premium reduces to \$336.81.

5 Bond

A simple bond (zero coupon bond) is a promise to pay the bond holder a cash amount (face value) at a future date (term of the bond). The price for the bond that the bond holder pays today is:

$$B_0 = \frac{B}{(1+r)^n} \tag{11}$$

where B_0 is the bond price, B bond face value, r the prevailing interest rate, and n the bond term (in years). The prevailing interest rate is typically the corresponding FED rate. A \$1,000, 10-year zero-coupon bond with interest rate of 3% can be purchased at \$744 today and get paid \$1000 10 years later.

Conventional bond also pays coupons on a periodic basis, monthly or quarterly. The price of the bond is then a combination of the discounted face value plus the discounted coupons. Since the bond is traded on the market every day for interest rate arbitrage, the prevailing interest rate is no longer a pre-determined value, but a variable one r_i.

$$B_0 = B \prod_{i=1}^{nm} \frac{1}{(1 + r_i/m)} + B \sum_{i=1}^{nm} \frac{c/m}{(1 + r_i/m)^i} \tag{12}$$

where m is the number of coupon payments in a given period (1 year) and c the coupon rate (as percentage of the the bond face value). As a simple illustration with $m = 1$, $r_i = r$, and $c = r$, B_0 is the same as B. There is no discount for the price since the coupon payment makes up the cash flow as the interest payment. However, if the coupon rate is less than the interest rate (e.g., monetary policy tightening) the bond becomes less valuable and the bond price deteriorates. Conversely, if the coupon rate is higher than the interest rate (e.g., monetary policy easing) the bond becomes more valuable and the bond price appreciates.

When the bond holder wishes to trade the bond before its maturity (at $i = k$ and $1 < k \leq nm$), the bond price at $i = k$ is:

$$B_k = B \prod_{i=k}^{nm} \frac{1}{(1 + r_i/m)} + B \sum_{i=k}^{nm} \frac{c/m}{(1 + r_i/m)^i} \tag{13}$$

6 Concluding Remarks

A promise made for future payments is discounted to today's money with the prevailing interest rate. This rate can be fixed by an initial contract or variable when traded on the open market for interest rate arbitrage. A simple interest rate (annualized) is used for most financial transactions when the underlying asset is evaluated and the interest compounded. More frequent compounding is also possible, e.g., monthly, bi-weekly, or daily. This approach is used for computing the present value of pension plans, annuity insurance products, consumer loans, and bonds that pay coupons. The monetary policy keenly affects the bond prices because bond is traded on the open market everyday. In a monetary easing environment, a bond becomes more valuable. Otherwise, in a monetary tightening environment, the bond price suffers.