<u>Black-Scholes Equation</u>
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The Black-Scholes equation solves the option price using the stochastic calculus which leads to the birth of modern finance. Understanding the Black-Scholes equation is an important part of today's economic science. This note is based on several mathematical lectures found on the Internet (*e.g.,* Michael Dabkowski of University of Michigan, Dearborn).

<u>Delta Hedging</u>

Delta hedging is a strategy that neutralizes a portfolio's sensitivity to stochastic price changes in the underlying asset by taking offsetting positions, i.e., buying or selling simultaneously the underlying stock and/or the related derivatives. Let P be a portfolio that buys one unit of call option contract (priced as C) and sells Δ shares of the underlying stock (priced as S):

$$P = C - \Delta S$$

The derivative of the portfolio is:

$$dP = dC - \Delta dS$$

A delta neutral portfolio should equal to another portfolio that only earns risk free interest, r:

$$dP = rPdt$$

Using the Ito's Lemma and $dS = \mu S dt + \sigma S dw$, noting dw as the Wiener process:

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}dt$$

Therefore,

$$dP = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right)dt + \left(\frac{\partial C}{\partial S} - \Delta\right)dS$$

If Δ is selected such that the second parenthesis term vanishes, the portfolio becomes risk free. This leads the portfolio to be the same as the second risk free portfolio:

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) = r\left(C - \frac{\partial C}{\partial S}S\right)$$

Rearrange the terms, one has the Black-Scholes equation for call option price, C:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

and the "initial condition" (expiration condition) as shown below, where K is the strike price of the option contract at expiration, T:

$C(S, t = T) = max(S - K, 0)$

<u>Solving the Partial Differential Equation</u>

The solution strategy is to transform this non-homogeneous, non-standard parabolic partial differential equation to a standard parabolic partial differential equation using variable substitutions. First, one can substitute the variables t and S using the relations below so that the "expiration condition" becomes the initial condition and the logarithmic stock price is used:

$$\tau = \frac{\sigma^2}{2}(T - t)$$
$$x = ln\left(\frac{S}{K}\right)$$

The partial derivatives of the option price, C, with respect to t and S become:

$$\frac{\partial C}{\partial t} = \frac{\partial C}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial C}{\partial \tau}$$

$$\frac{\partial C}{\partial S} = \frac{\partial C}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{S}\frac{\partial C}{\partial x}$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{1}{S^2}\left(\frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x}\right)$$

Now, the partial differential equation has been transformed in terms of variables τ (reverse the direction of time) and x (logarithmic stock price):

$$-\frac{\sigma^2}{2}\frac{\partial C}{\partial \tau} + r\frac{\partial C}{\partial x} + \frac{\sigma^2}{2}\left(\frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x}\right) = rC$$

and the initial condition has also been transformed to:

$$C(x, 0) = max\left(K(e^x - 1), 0\right)$$

Further, the constants r and σ^2 can be combined as a single λ:

$$\lambda = \frac{2r}{\sigma^2}$$

The transformed partial differential equation is much neater:

$$\frac{\partial C}{\partial \tau} = \frac{\partial^2 C}{\partial x^2} + (\lambda - 1)\frac{\partial C}{\partial x} - \lambda C$$

A mathematical trick is then used to eliminate the lower order terms in the equation by intentionally choosing the constants a and b in below function substitution using $u(x, \tau)$:

$C(x,\tau) = Ke^{ax+b\tau}u(x,\tau)$

After carrying out the tedious differentiations, the partial differential equation of $u(x,\tau)$ is:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (\lambda - 1 + 2a)\frac{\partial u}{\partial x} + \left[a(\lambda - 1) - \lambda + a^2 - b\right]u$$

Selecting the appropriate a and b such that all the lower order terms vanish:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

where $a = -(\lambda - 1)/2$ and $b = -(\lambda + 1)^2/4$. The initial condition of $u(x,\tau)$ (at $\tau = 0$) becomes:

$u(x,0) = max\left(e^{ax+b\tau}(e^x - 1), 0\right) = max\left(e^{(1-a)x} - e^{-ax}, 0\right)$, or

$$u(x,0) = max\left(e^{(\lambda+1)x/2} - e^{(\lambda-1)x/2}, 0\right)$$

<u>Applying the Initial Condition</u>

The solution of the standard parabolic partial differential equation with initial condition $u(x,0)$ is given by the following convolution integral:

$$u(x,\tau) = (\Gamma * u(x,0)) = \int_{-\infty}^{\infty} \Gamma(x - z)u(z,0)dz$$

where the kernel (filter) for the convolution is:

$$\Gamma(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}e^{-x^2/4\tau}$$

Since $u(z,0)$ has non-zero values only if $z > 0$, this is because z or $x = ln(S/K)$. If $S > K$ the payoff is positive, otherwise the payoff is zero at the contract expiration. Therefore, the integration of non-zero term for z is from 0 to ∞. Further, let $y = (z - x)/\sqrt{2\tau}$, the solution becomes:

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} u(x + \sqrt{2\tau}y, 0)e^{-y^2/2}dy$$

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} \left[e^{(\lambda+1)(x+\sqrt{2\tau}y)} - e^{(\lambda-1)(x+\sqrt{2\tau}y)}\right]e^{-y^2/2}dy$$

The next step is to rewrite the integrals in a form of the cumulative standard normal distribution function given the familiar term $exp(-y^2/2)dy$ in the integrals. Adding and subtracting $(\lambda + 1)^2\tau/2$ to the first exponential term, and adding and subtracting $(\lambda - 1)^2\tau/2$ to the second exponential term, the solution expands to:

$$u(x,\tau) = \frac{e^{(\lambda+1)x/2+(\lambda+1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-(\lambda+1)\sqrt{\tau/2})^2/2} dy$$

$$- \frac{e^{(\lambda-1)x/2+(\lambda-1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-(\lambda-1)\sqrt{\tau/2})^2/2} dy$$

Let $-v_1 = y - (\lambda+1)\sqrt{\tau/2}$ and $-v_2 = y - (\lambda-1)\sqrt{\tau/2}$. Please note the negative sign of these two new variables so that the upper and lower limits of the integrals are flipped:

$$u(x,\tau) = \frac{e^{(\lambda+1)x/2+(\lambda+1)^2\tau/4}}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{2\tau}+(\lambda+1)\sqrt{\tau/2}} e^{-v_1^2/2} dv_1$$

$$- \frac{e^{(\lambda-1)x/2+(\lambda-1)^2\tau/4}}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{2\tau}+(\lambda-1)\sqrt{\tau/2}} e^{-v_2^2/2} dv_2$$

Now, both integrals are the cumulative standard normal distribution function with the "z-scores" at their upper integral limits. Therefore,

$$u(x,\tau) = e^{(\lambda+1)x/2+(\lambda+1)^2\tau/4} N\left(\frac{x+(\lambda+1)\tau}{\sqrt{2\tau}}\right) - e^{(\lambda-1)x/2+(\lambda-1)^2\tau/4} N\left(\frac{x+(\lambda-1)\tau}{\sqrt{2\tau}}\right)$$

Back Substitution

Now, use $u(x,\tau) = C(x,\tau)/Ke^{ax+b\tau}$ where $a = -(\lambda-1)/2$ and $b = -(\lambda+1)^2/4$:

$$C(x,\tau) = Ke^{x} N\left(\frac{x+\lambda\tau+\tau}{\sqrt{2\tau}}\right) - Ke^{-\lambda\tau} N\left(\frac{x+\lambda\tau-\tau}{\sqrt{2\tau}}\right)$$

Note $\tau = \sigma^2(T-t)/2$ and $\lambda\tau = r(T-t)$; therefore

$$C(x,t) = Ke^{x} N\left(\frac{x+r(T-t)+\sigma^2(T-t)/2}{\sqrt{\sigma^2(T-t)}}\right) - Ke^{-r(T-t)} N\left(\frac{x+r(T-t)-\sigma^2(T-t)/2}{\sqrt{\sigma^2(T-t)}}\right)$$

Finally, substitute $e^x = S/K$, $x = ln(S/K)$, and reorganize the terms:

$$C(S,t) = SN\left(\frac{ln(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)} N\left(\frac{ln(S/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

where $t \in [0,T)$.