Binomial Theorem
02/25/2025

Binomial series expansion does not have direct applications in counting. However, the proof of the Binomial Theorem could involve combinatorics, or counting. Here is the Binomial Theorem:

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

Combinatorial Proof: $(x+y)^n$ is a product of n factors of $(x+y)$. As such, a term of the form $x^{n-j}y^j$ arises from choosing y from j factors of $(x+y)$ and x from the other $(n-j)$ factors of $(x+y)$. This can be done in nCj ways, since nCj counts the number of ways of selecting j items from n items. Therefore, $x^{n-j}y^j$ appears in the binomial expansion nCj times. $\square$

$$NOTE: \quad nCj \equiv \binom{n}{j}$$

Mathematical Induction Proof: Base Case, $n=0$:

$$(x+y)^0 = 1 = \binom{0}{0} x^{0-0} y^0$$

Induction Step,

$$(x+y)^{n+1} = (x+y)(x+y)^n$$
$$= \sum_{j=0}^{n} \binom{n}{j} x^{n+1-j} y^j + \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^{j+1}$$
$$= x^{n+1} + \sum_{j=1}^{n} \binom{n}{j} x^{n+1-j} y^j + \sum_{j=0}^{n-1} \binom{n}{j} x^{n-j} y^{j+1} + y^{n+1}$$
$$= x^{n+1} + \sum_{j=1}^{n} \binom{n}{j} x^{n+1-j} y^j + \sum_{j=1}^{n} \binom{n}{j-1} x^{n-(j-1)} y^j + y^{n+1}$$
$$= x^{n+1} + \sum_{j=1}^{n} \left(\binom{n}{j} + \binom{n}{j-1} \right) x^{n+1-j} y^j + y^{n+1}$$
$$= \binom{n+1}{0} x^{n+1} + \sum_{j=1}^{n} \binom{n+1}{j} x^{n+1-j} y^j + \binom{n+1}{n+1} y^{n+1}$$

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$$= \sum_{j=0}^{n+1} \binom{n+1}{j} x^{n+1-j} y^j \square$$

NOTE: In the step next to the last Pascal's Identity is utilized (see below).

Corollary 1.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

This is a direct result of the Binomial Theorem with x= 1 and y = 1.

Corollary 2.

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

This is another direct result of the Binomial Theorem with x= 1 and y = -1.

Corollary 3.

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

This again is another direct result of the Binomial Theorem with x= 1 and y = 2.

Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Combinatorial Proof: Let T be a set containing $n+1$ elements and x be one of the elements. S is a subset of T with $T - \{x\}$ elements. Further, there are $(n+1)Ck$ subsets of T containing k elements. However, a subset of T with k elements either contains x together with $k-1$ other elements of S, or contains k elements of S without x. There are $nC(k-1)$ subsets of the former and nCk subsets of the latter $\square$.

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Algebraic Proof:

$$\binom{n}{k-1} + \binom{n}{k} = \frac{kn!}{k(k-1)!(n-k+1)!} + \frac{(n-k+1)n!}{k!(n-k)!(n-k+1)}$$
$$= \frac{(n+1)n!}{k!(n-k+1)(n-k)!)} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k} \square$$

<u>Vandermonde's Identity</u>

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$$

Combinatorial Proof: There are m items in one set and n in the other. The total number of ways to pick r items from the union of these two sets is $(m+n)Cr$. Another approach to pick r items from the union of these two sets is to pick k items from the second set and $(n-k)$ items from the first set, where $0 \leq k \leq r$. There are nCk ways to choose k items from the second set and $mC(r-k)$ ways to choose $(r-k)$ items from the first set. Based on the product rule, there are $mC(r-k) \times nCk$ ways for each k. Using the addition rule, the total number of ways to pick r items from the union also equals:

$$\sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k} \square$$

<u>Corollary 4</u>.

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

This is a direct result of Vandermonde's Identity with m = n and r = n.

<u>An Unnamed Identity</u>.

$$\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$$
where $r \leq n$.

Combinatorial Proof: Let a bit string of $n+1$ in length contain $r+1$ ones. There are $(n+1)C(r+1)$ ways to construct such a bit string. Let's also consider the position of the last

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1-bit in the bit string. This last 1-bit must be located at position $r+1, r+2, \ldots$, or $n+1$. Further, if this last 1-bit is at the $k-th$ location, there must be r ones in the previous $k-1$ locations. There are $(k-1)Cr$ such bit strings. Using the addition rule over the k through $r+1 \leq k \leq n+1$, the number of such bit strings can also be computed as:

$$\sum_{k=r+1}^{n+1} \binom{k-1}{r} = \sum_{j=r}^{n} \binom{j}{r}$$

NOTE: index substitution of $j = k-1$ is made above $\Box$.

Concluding Remarks

In combinatorial proofs, the basic approach is to count a model set in several different ways. One can break the original model set into two or more subsets and then use the product rule and addition rule to count the same model from the subsets. The subsets can be as direct as the m and n elements used for proving Vandermode's Identity, or as convoluted as the one with or without a single element $\{x\}$ for proving Pascal's Identity. Constructing a bitstring is an interesting yet less convoluted idea.

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