Material Viscoelasticity
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Phenomenological models of viscoelasticity occupy a central place in the mechanical characterization of polymers, biological tissues, and amorphous metals. By postulating combinations of springs and dashpots, these models provide a tractable mathematical framework that captures the essential features of time-dependent deformation: creep, stress relaxation, rate sensitivity, and hysteresis. Among them, the *Maxwell* model and its generalization through the *Prony* series (often called the generalized *Maxwell* or *Wiechert* model) have become the most widely used representations, particularly in commercial finite element software and in the analysis of polymeric materials. Despite their extensive application, these models rest on a set of fundamental assumptions that impose severe limitations on their predictive capability. This article examines the conceptual foundations, governing equations, logical connections, and practical usage of the leading linear viscoelasticity models, with a critical focus on the *Maxwell* element and the *Prony* series representation. The analysis demonstrates that while these models offer a mathematically elegant and computationally efficient means to describe linear viscoelastic behavior, their inherent linearity, their reliance on discrete relaxation spectra, and their phenomenological character preclude an accurate description of nonlinearities, physical aging, and the complex molecular dynamics that govern real viscoelastic materials.

Assumptions of Linear Viscoelasticity

All classical phenomenological models of viscoelasticity are built upon the premise of linearity, i.e., that the stress at any time is a linear functional of the strain history, and conversely. This linearity assumption, formalized by *Boltzmann*'s superposition principle, is the cornerstone of the theory. It implies that the creep compliance $J(t)$ and relaxation modulus $G(t)$ are material functions independent of the applied stress or strain magnitude, and that the response to a complex history can be obtained by convolution. The second fundamental assumption is that of time-translational invariance: the material's properties do not change with age or with the rate of deformation, except through the explicit time dependence. For many engineering polymers and biomaterials operating under small deformations, these assumptions provide a reasonable approximation over limited time and temperature ranges. However, they inherently exclude the ubiquitous phenomena of strain-dependent relaxation times, physical aging, and nonlinearities that arise when molecular chains undergo large orientation or when filler networks break down.

Maxwell Model: The Simplest Differential Formulation

The Maxwell model is constructed from a linear spring (elastic modulus E) and a linear dashpot (viscosity η) connected in series. Its fundamental assumption is that the total strain is the sum of an elastic and a viscous contribution, $\varepsilon = \varepsilon_e + \varepsilon_v$, with $\sigma = E\varepsilon_e = \eta\dot{\varepsilon}_v$. The governing differential equation is derived by differentiating the strain decomposition and substituting the constitutive relations:

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}$$

This single-element model captures stress relaxation at constant strain ($\dot{\varepsilon} = 0$, $\sigma(t) = \sigma_0 e^{-t/\tau}$ and $\tau = \eta/E$) and creep at constant stress ($\dot{\sigma} = 0$, $\varepsilon(t) = \sigma_0(1/E + t/\eta)$). Its logical derivation follows directly from the rheological diagram and the assumption that the elements are subject to the same stress. The model is mathematically simple, requiring only two parameters.

The critical shortcomings of the *Maxwell* model are, however, profound. Its creep response is linear in time without a bounded steady-state compliance, which is unrealistic for cross-linked polymers and many biological tissues. Conversely, its relaxation function is a single exponential, whereas real materials display a broad distribution of relaxation times. The model also fails to capture the glassy response at short times and the rubbery plateau at long times. As a result, the *Maxwell* model is seldom used alone for quantitative material modeling; its principal value lies in its role as a building block for more sophisticated representations, particularly the generalized *Maxwell* model.

Generalized Maxwell Model and the Prony Series

To overcome the limitations of a single relaxation time, the generalized *Maxwell* model (*Wiechert* model) arranges an arbitrary number of *Maxwell* elements in parallel, often with an additional free spring to represent the equilibrium modulus. The fundamental assumption is that the total stress is the sum of the stresses in each branch, all of which experience the same strain history. The relaxation modulus then takes the form of a *Prony* series:

$$G(t) = G_\infty + \sum_{i=1}^{N} G_i \, e^{-t/\tau_i}$$

where G_∞ is the long-term (equilibrium) modulus, G_i are the relaxation strengths, and $\tau_i = \eta_i / G_i$ are the discrete relaxation times. The corresponding creep compliance can be expressed, under the assumption of linearity, as a *Dirichlet* series. The governing equations in the time domain are usually written as a set of ordinary differential equations for the internal strains or stresses in each branch, or, more conveniently, as a convolution integral using the relaxation kernel $G(t)$. In the frequency domain, the complex modulus is given by

$$G^*(\omega) = G_\infty + \sum_{i=1}^{N} G_i \frac{i\omega\tau_i}{1 + i\omega\tau_i}$$

which directly connects to dynamic mechanical analysis (DMA) data.

The logical derivation of the *Prony* series is either phenomenological—fitting the relaxation spectrum with a discrete set of exponentials—or motivated by the molecular theory of polymer dynamics, where the relaxation times correspond to different modes of chain motion. In practice, the parameters G_i and τ_i are identified by least-squares fitting to experimental data from relaxation, creep, or DMA tests. The series can be made arbitrarily accurate by increasing N, provided the data span a sufficiently wide time or frequency range. To construct a master curve spanning an extended frequency range, the time-temperature superposition principle is invoked. This principle asserts that the viscoelastic response at a reference temperature T_{ref} can be related to that at another temperature T through a horizontal shift factor a_T. The reduced frequency is defined as $\omega_r = a_T \omega$. The resulting shift factors a_T as a function of temperature are then fitted to the *Williams-Landel-Ferry* (WLF) equation:

$$\log_{10}(a_T) = \frac{-C_1 (T - T_{\mathrm{ref}})}{C_2 + (T - T_{\mathrm{ref}})}$$

Once the shift factors are determined, the reduced frequency master curve is assembled by horizontally shifting each isothermal dataset to the reference temperature. The combined dataset now represents the material's viscoelastic response over an extended reduced frequency range, often spanning 10 to 15 decades.

Critical Analysis of the Maxwell-Prony Framework

The most fundamental limitation is the assumption of linearity. Real viscoelastic materials, especially polymers above their glass transition temperature or under large strains, exhibit pronounced nonlinearities. The relaxation time becomes strain-dependent, the *Boltzmann* superposition principle fails, and the creep compliance is no longer a function of time alone. The *Prony* series, being a linear model, cannot capture such behavior without ad-hoc modifications such as the introduction of nonlinear spring elements or the use of a "quasi-linear" viscoelasticity framework. In many engineering applications, the linear assumption is retained even when deformations exceed the linear range, leading to significant predictive errors.

A second criticism concerns the physical interpretation of the discrete relaxation spectrum. The parameters G_i and τ_i are purely phenomenological; there is no unique mapping from a real material's continuous relaxation spectrum to a finite *Prony* series. Different numerical optimization algorithms can yield vastly different sets of parameters that fit the same data equally well, especially when the number of terms N is chosen arbitrarily. This non-uniqueness undermines the predictive reliability of the model for loading histories that differ from the calibration test. Moreover, the choice of N is often governed by convenience rather than by physical insight: too few terms lead to poor fits, while too many terms cause overfitting and numerical ill-conditioning in finite element implementations.

A third issue lies in the time-temperature superposition principle (TTSP), which assumes that the relaxation modulus at different temperatures can be shifted along the logarithmic time axis by a shift factor $a_T(T)$, *i.e.*, $G(t,T) = G(t/a_T, T_0)$. While TTSP is remarkably successful for amorphous polymers in the glass-rubber transition region, it is an empirical approximation that fails for semi-crystalline polymers, composites, and materials undergoing physical aging or chemical degradation. The *Prony* series combined with TTSP therefore inherits all the limitations of that principle.

Another practical drawback is the computational expense when a large number of Maxwell branches is used in transient finite element simulations. Each branch introduces an additional internal state variable, and the time integration must resolve the smallest relaxation time, which can lead to stiff systems. Although modern implicit integration schemes mitigate this, the complexity grows with N. Furthermore, the *Prony* series is inherently a small-strain formulation; extending it to finite deformations requires careful treatment of objectivity and the choice of appropriate strain measures, often leading to formulations that are no longer rigorously derived from thermodynamics but rather from ad-hoc generalizations.

Alternative Representations and Their Comparative Merits

To address the deficiencies of the *Maxwell-Prony* framework, several alternative phenomenological models have been developed. The *Kelvin-Voigt* model (spring and dashpot in parallel) gives a bounded creep compliance but an unrealistic instantaneous response and does not relax stress fully. The standard linear solid (SLS), which combines a *Maxwell* element in parallel with a spring, introduces a single relaxation time and an equilibrium modulus, but still lacks the spectral breadth of real materials. More sophisticated fractional calculus models replace integer-order derivatives with fractional ones, yielding a continuous relaxation spectrum with fewer parameters. While these models offer a more parsimonious representation, they have not achieved the same level of adoption in commercial finite element codes as the *Prony* series, partly because of implementation complexity and the unfamiliarity of fractional operators to many engineers.

In actual usage, the *Prony* series remains the dominant representation for linear viscoelasticity in industry. It is implemented in virtually all major finite element packages (Abaqus, Ansys, Nastran) and is the standard for modeling polymers, composites, and biological tissues under small deformations. The calibration process is well established: DMA or relaxation data are fitted to obtain the *Prony* coefficients, and the shift factors are determined from master curve construction. For many practical problems, such as the prediction of residual stresses in molded parts, the damping of polymer components, or the time-dependent deformation of asphalt pavements, the *Prony* series provides acceptable accuracy when used within the linear regime. However, users must be acutely aware that the model's predictions for nonlinear, large-strain, or long-term aging scenarios are not reliable unless the model has been specifically extended and validated for those conditions.

Concluding Remarks

The leading phenomenological models of viscoelasticity—exemplified by the *Maxwell* element and its generalization via the *Prony* series—offer a mathematically coherent and computationally accessible means to describe time-dependent material behavior. Their fundamental assumptions of linearity, time invariance, and the separability of time and temperature effects allow them to be parameterized from standard laboratory tests and integrated into large-scale simulations. The *Maxwell* model, while too simplistic for most applications, provides the essential building block for the *Prony* series, which captures a broad distribution of relaxation times and reproduces the main features of linear viscoelastic response.

Nevertheless, a critical evaluation reveals that these models are constrained by the very assumptions that make them convenient. The linearity assumption is violated in most real materials when deformations exceed a few percent or when the material is subjected to large strains, high stresses, or complex multiaxial loading. The discrete *Prony* series suffers from non-uniqueness and overparameterization, often yielding a set of relaxation times and strengths that have no direct physical meaning and may not extrapolate reliably. The reliance on time-temperature superposition restricts applicability to a narrow class of amorphous polymers under controlled conditions. Moreover, the framework inherently neglects nonlinearities such as strain-dependent relaxation, damage accumulation, and physical aging, all of which are critical for accurate long-term predictions.

In actual engineering practice, the *Prony* series is used extensively, but often without a critical appraisal of its limitations. The temptation to treat the fitted parameters as intrinsic material constants, and to apply the model outside the range of calibration, is a common source of error. The future development of viscoelastic modeling lies in embracing nonlinear frameworks—such as the *Schapery* model, nonlinear fractional viscoelasticity, or physics-based molecular network models—that retain computational tractability while relaxing the restrictive assumptions of linearity and spectral discreteness. Until such alternatives become as widespread as the *Prony* series, engineers must apply the current phenomenological models with a clear understanding of their foundational assumptions and a healthy skepticism regarding their predictive reach.