Tensor Calculus
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The calculus of vectors and matrices, while powerful, is insufficient for describing physical quantities whose transformation properties under changes of coordinate system are more complex than those of ordinary vectors. Tensor calculus provides a unified language capable of expressing physical laws in a form that is *independent of the choice of coordinate system* — a principle sometimes called *covariance* or *form-invariance*. The fundamental assumptions of tensor calculus are as follows.

Assumption 1 (Underlying manifold). The physical or mathematical space of interest is a smooth *manifold* $\mathcal{M}$ of dimension n — a set that locally resembles $\mathbb{R}^n$ and on which smooth coordinate charts can be defined. In many applications, $\mathcal{M} = \mathbb{R}^n$ with Cartesian coordinates, but the theory accommodates curvilinear coordinates and curved spaces.

Assumption 2 (Coordinate transformation). At each point $p \in \mathcal{M}$, one may choose coordinates $\{x^i\}$ or alternatively $\{\bar{x}^i\}$, related by a smooth, invertible transformation $\bar{x}^i = \bar{x}^i(x^1, \ldots, x^n)$. The *Jacobian* of this transformation is

$$J^i{}_j = \frac{\partial \bar{x}^i}{\partial x^j}, \qquad \tilde{J}^i{}_j = \frac{\partial x^i}{\partial \bar{x}^j} \qquad \text{with } J^i{}_k \tilde{J}^k{}_j = \delta^i{}_j.$$

Assumption 3 (Tensor transformation law). A *tensor of type* (r, s) (also called a tensor of *contravariant order* r and *covariant order* s, or simply an (r, s)-tensor) at a point p is a multilinear map that assigns a real number to r covectors (dual vectors, or one-forms) and s vectors. Its components transform under a change of coordinates by

$$\bar{T}^{i_1 \cdots i_r}{}_{j_1 \cdots j_s} = J^{i_1}{}_{k_1} \cdots J^{i_r}{}_{k_r} \, \tilde{J}^{l_1}{}_{j_1} \cdots \tilde{J}^{l_s}{}_{j_s} \, T^{k_1 \cdots k_r}{}_{l_1 \cdots l_s}$$

This transformation law is the defining property of a tensor. Quantities satisfying it are called *tensors*; those that do not are not tensors in the strict sense (*e.g.*, *Christoffel* symbols, discussed below).

Assumption 4 (Index notation conventions). Upper indices are *contravariant* (transforming with the forward *Jacobian* J) and lower indices are *covariant* (transforming with the inverse *Jacobian* $\tilde{J}$). In Euclidean space with orthonormal coordinates, the distinction vanishes and all indices may be written as subscripts.

Assumption 5 (Metric tensor). A *Riemannian* manifold is equipped with a *metric tensor* g_{ij}, a symmetric, positive-definite $(0, 2)$-tensor that provides a notion of distance and angle. In Cartesian coordinates, $g_{ij} = \delta_{ij}$. The metric tensor and its inverse g^{ij} (satisfying $g^{ik} g_{kj} = \delta^i_j$) serve to raise and lower indices:

$$v_i = g_{ij} v^j, \qquad v^i = g^{ij} v_j$$

Classification of Tensors

A scalar is a $(0, 0)$-tensor. A contravariant vector (or simply a vector) is a $(1, 0)$-tensor, with components v^i transforming as $\bar{v}^i = J^i{}_j v^j$. A *covariant vector* (or *covector*, or *one-form*) is a $(0, 1)$-tensor, with components ω_i transforming as $\bar{\omega}_i = \tilde{J}^j{}_i \omega_j$. A *second-order tensor* (or rank-2 tensor) can be of type

$(2, 0)$, $(1, 1)$, or $(0, 2)$, represented by components T^{ij}, $T^i{}_j$, or T_{ij}, respectively. In Euclidean space, the matrix $\mathbf{A}$ is a $(1, 1)$-tensor (or equivalently a $(0, 2)$-tensor when the metric is used to lower one index). The stress tensor in mechanics, the inertia tensor in rigid body dynamics, and the electromagnetic field tensor in electrodynamics are all examples of second-order tensors. Higher-order tensors, such as the *elasticity tensor* C_{ijkl} (a $(0, 4)$-tensor), arise naturally in the constitutive theory of materials.

Algebraic Operations

Addition and Scaler Multiplication
Two tensors of the same type (r, s) may be added component-wise. The result is a tensor of the same type. In both vector and component notation:

$$\mathbf{T} + \mathbf{S} \text{ (intrinsic)}, \qquad (T+S)^{i_1 \cdots i_r}{}_{j_1 \cdots j_s} = T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s} + S^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}$$

Scalar multiplication follows analogously:

$$\alpha \mathbf{T} \text{ (intrinsic)}, \qquad (\alpha T)^{i_1 \cdots i_r}{}_{j_1 \cdots j_s} = \alpha \, T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}$$

The tensor type must match for addition to be defined.

Tensor (Outer) Product
Given a tensor $\mathbf{T}$ of type (r_1, s_1) and a tensor $\mathbf{S}$ of type (r_2, s_2), their tensor product $\mathbf{T} \otimes \mathbf{S}$ is a tensor of type $(r_1 + r_2, s_1 + s_2)$, defined by

$$(\mathbf{T} \otimes \mathbf{S})^{i_1 \cdots i_{r_1} i_{r_1+1} \cdots i_{r_1+r_2}}{}_{j_1 \cdots j_{s_1} j_{s_1+1} \cdots j_{s_1+s_2}} = T^{i_1 \cdots i_{r_1}}{}_{j_1 \cdots j_{s_1}} \cdot S^{i_{r_1+1} \cdots i_{r_1+r_2}}{}_{j_{s_1+1} \cdots j_{s_1+s_2}}$$

In particular, the tensor product of two vectors $\mathbf{u}$ and $\mathbf{v}$ gives a $(2, 0)$-tensor:

$$(\mathbf{u} \otimes \mathbf{v})^{ij} = u^i v^j$$

which in matrix terms corresponds to the outer product $\mathbf{u}\mathbf{v}^T$.

Contraction
Contraction is the operation that reduces a tensor of type (r, s) (with $r \geq 1$ and $s \geq 1$) to type $(r-1, s-1)$ by setting one contravariant index equal to one covariant index and summing. For a $(1, 1)$-tensor, contraction yields the trace:

$$T^i{}_i = \mathrm{tr}(\mathbf{T})$$

For a $(2, 1)$-tensor $T^{ij}{}_k$, contraction on the first contravariant and the covariant index gives $T^{ij}{}_j$, which is a contravariant vector.

Inner (Dot) Product via Contraction
The contraction of a $(1, 0)$-tensor u^i with a $(0, 1)$-tensor ω_j yields the scalar:

$$\langle \boldsymbol{\omega}, \mathbf{u} \rangle = \omega_i u^i$$

The contraction of two $(0,2)$-tensors $\mathbf{A}$ and $\mathbf{B}$ with one shared index gives matrix multiplication:

$$(\mathbf{AB})_{ij} = A_{ik}B_{kj}$$

Contraction with both shared indices yields the scalar *Frobenius* product $A_{ij}B_{ij} = \mathbf{A} : \mathbf{B}$.

Symmetrization and Antisymmetrization
Given any $(0,2)$-tensor T_{ij}, it may be decomposed into its symmetric part

$$T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji})$$

and its antisymmetric part

$$T_{[ij]} = \frac{1}{2}(T_{ij} - T_{ji})$$

therefore $T_{ij} = T_{(ij)} + T_{[ij]}$. These operations generalize to higher-order tensors.

Derivatives of Tensors
In flat Euclidean space with Cartesian coordinates, the partial derivative of a tensor field $T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}(\mathbf{x})$ with respect to x^k produces a tensor of type $(r, s+1)$, written $\partial_k T^{i_1 \cdots i_r}{}_{j_1 \cdots j_s}$.

However, in curvilinear or curved coordinates, partial differentiation does not preserve tensorial character. The correct generalization is the *covariant derivative*, which accounts for the variation of the basis vectors from point to point.

Covariant Derivative
The *Christoffel* symbols $\Gamma^k{}_{ij}$ (also called the connection coefficients) encode how the basis vectors change from point to point. They are defined by

$$\Gamma^k{}_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij} \right)$$

and are symmetric in i and j: $\Gamma^k{}_{ij} = \Gamma^k{}_{ji}$. Note that the *Christoffel* symbols are not tensors, as they do not obey the tensor transformation law.

The covariant derivative of a contravariant vector v^i is defined as

$$\nabla_k v^i = \partial_k v^i + \Gamma^i{}_{kj} v^j$$

and that of a covariant vector ω_i as

$$\nabla_k \omega_i = \partial_k \omega_i - \Gamma^j{}_{ki} \omega_j$$

The sign change between the two cases reflects the dual transformation properties of contravariant and covariant indices. For a general (r,s)-tensor, the covariant derivative introduces one correction term of the form $+\Gamma^{i_a}{}_{kl} T^{\cdots l \cdots}{}_{\cdots}$ for each contravariant index i_a and one term of the form $-\Gamma^l{}_{kj_b} T^{\cdots}{}_{\cdots l \cdots}$ for each covariant index j_b.

In Cartesian coordinates, all *Christoffel* symbols vanish and $\nabla_k \equiv \partial_k$.

Addition Rule for Covariant Derivatives

$$\nabla_k(\mathbf{T} + \mathbf{S}) = \nabla_k \mathbf{T} + \nabla_k \mathbf{S}$$

in components,

$$\nabla_k({T^i}_j + {S^i}_j) = \nabla_k {T^i}_j + \nabla_k {S^i}_j$$

Leibniz Rule (Product Rule) for Covariant Derivatives
The covariant derivative satisfies the *Leibniz rule* with respect to the tensor product:

$$\nabla_k(\mathbf{T} \otimes \mathbf{S}) = (\nabla_k \mathbf{T}) \otimes \mathbf{S} + \mathbf{T} \otimes (\nabla_k \mathbf{S})$$

in components, for a $(1,0) \otimes (1,0)$ example:

$$\nabla_k(u^i v^j) = (\nabla_k u^i) v^j + u^i (\nabla_k v^j)$$

For contraction (inner product), the rule takes the form:

$$\nabla_k(v^i \omega_i) = (\nabla_k v^i)\omega_i + v^i(\nabla_k \omega_i) = \partial_k(v^i \omega_i)$$

showing that the covariant derivative of the scalar $v^i \omega_i$ equals its ordinary partial derivative, as required.

Compatibility with the Metric

The *Levi-Civita* connection is the unique torsion-free connection satisfying metric compatibility:

$$\nabla_k g_{ij} = 0, \qquad \nabla_k g^{ij} = 0$$

This ensures that raising and lowering of indices commutes with covariant differentiation:

$$\nabla_k v_i = g_{ij} \nabla_k v^j$$

The Covariant Gradient, Divergence, and Laplacian
In tensor calculus, the operators of vector calculus generalize as follows. The gradient of a scalar field ϕ is the covariant vector

$$(\nabla \phi)_i = \nabla_i \phi = \partial_i \phi$$

The divergence of a contravariant vector field v^i is the scalar

$$\nabla_i v^i = \partial_i v^i + {\Gamma^i}_{ij} v^j = \frac{1}{\sqrt{g}} \partial_i(\sqrt{g}\, v^i)$$

where $g = \det(g_{ij})$. The covariant *Laplacian* (or *Laplace–Beltrami* operator) is

$$\Delta\phi = \nabla_i \nabla^i \phi = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g}\, g^{ij} \partial_j \phi \right)$$

The Lie Derivative
An alternative derivative operation, the *Lie derivative* $\mathcal{L}_{\mathbf{v}}\mathbf{T}$, measures the change of a tensor field $\mathbf{T}$ along the flow of a vector field $\mathbf{v}$. For a $(0,2)$-tensor:

$$(\mathcal{L}_{\mathbf{v}} T)_{ij} = v^k \partial_k T_{ij} + T_{kj} \partial_i v^k + T_{ik} \partial_j v^k$$

The Lie derivative is intrinsic to the manifold structure and does not require a connection.

The Riemann Curvature Tensor
The failure of covariant derivatives to commute defines the *Riemann* curvature tensor $R^l{}_{kij}$, a $(1,3)$-tensor:

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) v^l = R^l{}_{kij} v^k$$

In components:

$$R^l{}_{kij} = \partial_i \Gamma^l{}_{jk} - \partial_j \Gamma^l{}_{ik} + \Gamma^l{}_{im} \Gamma^m{}_{jk} - \Gamma^l{}_{jm} \Gamma^m{}_{ik}$$

In flat (Euclidean) space, $R^l{}_{kij} = 0$ identically. The curvature tensor plays a central role in general relativity.

The Divergence Theorem and Stokes' Theorem in Tensor Form
Let $\Omega \subset \mathcal{M}$ be a domain with smooth boundary $\partial\Omega$ and outward unit normal n_i. *The Gauss divergence theorem* states

$$\int_\Omega \nabla_i T^{ij_1 \cdots j_s}{}_{k_1 \cdots k_r} \, dV = \oint_{\partial\Omega} n_i \, T^{ij_1 \cdots j_s}{}_{k_1 \cdots k_r} \, dA$$

where $dV = \sqrt{g}\, dx^1 \cdots dx^n$ is the volume element. This theorem underpins the derivation of balance laws in continuum mechanics.

Concluding Remarks

Tensor calculus is not merely an abstract generalization of vector and matrix analysis; it is the natural language for any science that deals with quantities defined in space and their rates of change. The systematic introduction of the transformation law, the covariant derivative, and the curvature tensor provides tools that are simultaneously applicable to classical mechanics, relativistic physics, the geometry of neural network loss landscapes, and the structural analysis of deformable solids. Mastery of the component notation — with its summation convention, index gymnastics, and product rules — together with the intrinsic notation, equips the practitioner with a dual perspective that is both computationally effective and geometrically illuminating.