Limits, Continuity, and Function

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The central philosophical commitment of analysis is rigor: every theorem must follow by strict logical deduction from explicitly stated axioms. This stands in contrast to the intuitive, geometric reasoning of early calculus, which, while powerful, concealed subtleties that led to paradoxes and errors. The 19th-century program of arithmetization of analysis — carried out principally by Cauchy, Weierstrass, Dedekind, and Cantor — replaced geometric intuition with precise ε-δ definitions.

Real Number System

The real numbers $\mathbb{R}$ are characterized by three families of axioms:

$\mathbb{R}$ is a *field* — a set equipped with addition $+$ and multiplication $\times$ satisfying commutativity, associativity, distributivity, and the existence of identity elements 0 and 1 and of additive and multiplicative inverses. Formally, for all $a, b, c \in \mathbb{R}$:

$$a + (b + c) = (a + b) + c, \qquad a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$\mathbb{R}$ is an *ordered field*: there exists a subset $\mathbb{R}^+ \subset \mathbb{R}$ (the positive reals) such that $\mathbb{R}^+$ is closed under addition and multiplication, and every $x \in \mathbb{R}$ satisfies exactly one of $x \in \mathbb{R}^+$, either $x = 0$, or $-x \in \mathbb{R}^+$ (the trichotomy law). These axioms give meaning to $<, \leq, >$, and $\geq$.

The *Completeness Axiom* (Least Upper Bound Property). This is the axiom that distinguishes $\mathbb{R}$ from $\mathbb{Q}$ (the rationals). It states:

Every non-empty subset $S \subseteq \mathbb{R}$ that is bounded above has a least upper bound (or *supremum*), denoted $\sup S \in \mathbb{R}$. A set S is bounded above if there exists $M \in \mathbb{R}$ such that $s \leq M$ for all $s \in S$. The supremum $\sup S$ is the smallest such M. Analogously, the *infimum* $\inf S$ is the greatest lower bound.

The equation $x^2 = 2$ has no solution in $\mathbb{Q}$ — the set $\{q \in \mathbb{Q} : q^2 < 2\}$ is bounded above in $\mathbb{Q}$ but has no rational least upper bound. Completeness guarantees that $\sqrt{2} \in \mathbb{R}$, plugging the "gaps" in the rationals. From the completeness axiom flow several foundational results:

The Archimedean Property. For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$. Equivalently, $\mathbb{N}$ is not bounded above in $\mathbb{R}$. This rules out "infinitely large" real numbers and justifies integer-based approximation.

Density of $\mathbb{Q}$ *in* $\mathbb{R}$. Between any two distinct reals $a < b$, there exists a rational $q \in \mathbb{Q}$ with $a < q < b$. This means every real number can be approximated arbitrarily closely by rationals — the basis of all numerical computation.

Sequences and Convergence

A sequence is a function $f : \mathbb{N} \to \mathbb{R}$, whose values are written $a_1, a_2, a_3, \ldots$ or $(a_n)_{n=1}^{\infty}$.

A sequence (a_n) converges to a limit $L \in \mathbb{R}$, written $\lim\limits_{n \to \infty} a_n = L$, if:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n > N \implies |a_n - L| < \varepsilon$$

No matter how small a "tolerance" ε one demands, all but finitely many terms of the sequence lie within ε of L. A sequence that does not converge is said to diverge.

A sequence (a_n) is a *Cauchy sequence* if its terms become mutually arbitrarily close:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } m, n > N \implies |a_m - a_n| < \varepsilon$$

Theorem (Cauchy's Completeness Criterion). A sequence of real numbers converges if and only if it is a Cauchy sequence.

This theorem is a direct consequence of the completeness of $\mathbb{R}$. Its power is that one can verify convergence without knowing the limit in advance — a crucial tool in analysis and numerical methods alike.

Theorem. Every bounded monotone sequence converges.

More precisely: if (a_n) is non-decreasing ($a_{n+1} \geq a_n$ for all n) and bounded above, then $\lim\limits_{n \to \infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$.

This theorem underlies the convergence proofs for many iterative algorithms in engineering, such as fixed-point iterations.

Theorem. Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

A subsequence (a_{n_k}) is obtained by selecting an infinite number of terms from (a_n) in order. This theorem is topological in character and underpins compactness arguments throughout analysis.

Limits and Continuity

Let $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$, and let c be a limit point (or accumulation point) of D — a point every neighborhood of which contains a point of D other than itself. The limit of f at c is L, written $\lim\limits_{x \to c} f(x) = L$, if:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$$

The condition $0 < |x - c|$ (strict positivity) means the value $f(c)$ itself is irrelevant to the limit.

A function $f : D \to \mathbb{R}$ is continuous at $c \in D$ if $\lim\limits_{x \to c} f(x) = f(c)$, which combines the existence of the limit, its equality to $f(c)$, and the definition of $f(c)$ at the point. Equivalently:

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon$$

f is continuous on D if it is continuous at every point of D.

Uniform continuity strengthens this by requiring δ to depend only on ε, not on c:

$$\forall\varepsilon > 0,\ \exists\delta > 0 \text{ such that } \forall x, y \in D,\ |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Extreme Value Theorem. If f is continuous on a closed bounded interval $[a, b]$, then f attains its maximum and minimum values.

That is, $\exists\, x_{\max}, x_{\min} \in [a, b]$ such that $f(x_{\min}) \leq f(x) \leq f(x_{\max})$ for all $x \in [a, b]$. The hypothesis that $[a, b]$ be closed (includes endpoints) and bounded (finite length) is essential — neither can be dropped.

Intermediate Value Theorem (IVT). If f is continuous on $[a, b]$ and k is any value strictly between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ with $f(c) = k$.

Formally: if $f(a) < k < f(b)$, then $\exists\, c \in (a, b)$ such that $f(c) = k$. This theorem expresses the connectedness of intervals and is used to prove the existence of roots (*e.g.*, bisection method convergence).

Heine–Cantor Theorem. A continuous function on a closed bounded interval $[a, b]$ is uniformly continuous.

Infinite Series

Given a sequence (a_n), the infinite series $\displaystyle\sum_{n=1}^{\infty} a_n$ is defined as the limit of its partial sums

$$S_N = \sum_{n=1}^{N} a_n: \qquad \sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$$

provided this limit exists (in which case the series *converges).

Necessary condition for convergence: If $\sum a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$. The converse is false (as the harmonic series demonstrates).

Comparison Test. If $0 \leq a_n \leq b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ converges.

Ratio Test. Let $L = \lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right|$. If $L < 1$, the series converges absolutely; if $L > 1$, it diverges; if $L = 1$, the test is inconclusive.

Root Test. Let $L = \limsup\limits_{n\to\infty} |a_n|^{1/n}$. The same conclusions hold as in the ratio test.

Integral Test. If f is positive, continuous, and decreasing on $[1, \infty)$ and $a_n = f(n)$, then $\sum a_n$ and $\displaystyle\int_1^{\infty} f(x)\, dx$ either both converge or both diverge.

A series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges. Absolute convergence implies convergence, but not vice versa. Conditionally convergent series (convergent but not absolutely convergent) are sensitive to rearrangement — the Riemann Rearrangement Theorem states that by reordering terms one can make such a series converge to any real number, or diverge.

Power Series

A power series centered at a is an infinite series of the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

Every power series has a radius of convergence $R \in [0, +\infty]$, given by:

$$\frac{1}{R} = \limsup_{n\to\infty} |c_n|^{1/n}$$

The series converges absolutely for $|x-a| < R$ and diverges for $|x-a| > R$. Within its interval of convergence, a power series defines a continuous, infinitely differentiable function, and can be differentiated and integrated term by term.

Sequences and Series of Functions

Let (f_n) be a sequence of functions on a set D. The sequence converges pointwise to f if, for each $x \in D$:

$$\lim_{n\to\infty} f_n(x) = f(x)$$

The sequence converges uniformly to f if:

$$\lim_{n\to\infty} \sup_{x\in D} |f_n(x) - f(x)| = 0$$

equivalently, $\forall \varepsilon > 0,\ \exists N$ (independent of x) such that $n > N \implies |f_n(x) - f(x)| < \varepsilon$ for all $x \in D$.

Uniform convergence is strictly stronger than pointwise convergence and has fundamentally better properties.

Theorem. If $f_n \to f$ uniformly on $[a, b]$ and each f_n is continuous, then f is continuous and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

This justifies the interchange of limit and integral — an operation that fails under mere pointwise convergence.
