Complex Calculus

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Let $f : U \to \mathbb{C}$ where $U \subseteq \mathbb{C}$ is open. The function f is complex differentiable (or holomorphic) at $z_0 \in U$ if the following limit exists:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}, \qquad h \in \mathbb{C},\ h \neq 0$$

The critical distinction from real differentiability is that $h \to 0$ in $\mathbb{C}$, so h may approach zero from any direction in the plane. Requiring the limit to be independent of direction is an enormously strong constraint.

If f is holomorphic at every point of U, then f is called *holomorphic* on U. A function holomorphic on all of $\mathbb{C}$ is called *entire*.

The Cauchy–Riemann Equations

Let $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$ and $u, v : \mathbb{R}^2 \to \mathbb{R}$ are the real and imaginary parts of f.

Theorem (Cauchy–Riemann). If f is holomorphic at $z_0 = x_0 + iy_0$, then the partial derivatives of u and v exist at (x_0, y_0) and satisfy:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These are the *Cauchy–Riemann (C–R) equations*. The derivative is then given by:

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Logical justification. Approaching z_0 along the real direction ($h = \Delta x$) yields:

$$f'(z_0) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Approaching along the imaginary direction ($h = i\Delta y$) yields:

$$f'(z_0) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Equating real and imaginary parts of these two expressions produces exactly the C–R equations.

Converse (sufficient conditions). If u and v have continuous partial derivatives on U satisfying the C–R equations, then $f = u + iv$ is holomorphic on U.

Harmonic Functions. Taking the divergence of the C–R equations reveals a profound consequence. If $f = u + iv$ is holomorphic and $u, v \in C^2$, then:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Both u and v satisfy *Laplace's equation* $\Delta\phi = 0$, and are called *harmonic functions*. They are said to be harmonic conjugates of each other. This connects complex analysis directly to potential theory: every holomorphic function generates a pair of harmonic functions, representing (for instance) velocity potentials and stream functions in fluid dynamics.

Complex Integration

A contour (or path) in $\mathbb{C}$ is a piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$, written $\gamma(t) = x(t) + iy(t)$. The contour integral of f along γ is defined as:

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

This is a line integral in the complex plane. The *ML-inequality* (modulus bound) provides an essential estimate:

$$\left|\int_\gamma f(z)\,dz\right| \leq \max_{z\in\gamma} |f(z)| \cdot \ell(\gamma)$$

where $\ell(\gamma) = \displaystyle\int_a^b |\gamma'(t)|\,dt$ is the arc length of γ.

Theorem (Cauchy's Integral Theorem). Let f be holomorphic on a simply connected domain $U \subseteq \mathbb{C}$, and let γ be any closed contour in U. Then:

$$\oint_\gamma f(z)\,dz = 0$$

A domain is simply connected if it has no holes — every closed curve in it can be continuously shrunk to a point. The theorem asserts that holomorphic functions have path-independent integrals, a direct analogue of conservative vector fields in $\mathbb{R}^2$ — which is no coincidence: the C–R equations are precisely the condition that $f(z)\,dz$ is an exact differential form.

Proof sketch. If $f \in C^1(U)$, the result follows from *Green's theorem*:

$$\oint_\gamma f\,dz = \iint_D \left(\frac{\partial f}{\partial \bar{z}}\right) d\bar{z} \wedge dz$$

where

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

is the *Wirtinger derivative*, which vanishes precisely when the C–R equations hold.

Theorem (Cauchy's Integral Formula). Let f be holomorphic on and inside a positively oriented (counterclockwise) simple closed contour γ. For any z_0 inside γ:

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz$$

This is one of the most remarkable theorems in mathematics: the values of a holomorphic function inside a region are completely determined by its values on the boundary. This rigidity — the global being determined by the local boundary — has no real analogue.

Differentiated form. Taking formal derivatives with respect to z_0 (justified by the dominated convergence theorem applied to the integral):

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

Corollary. Every holomorphic function is infinitely differentiable (smooth). This stands in stark contrast to real analysis, where a function may be once differentiable but not twice.

Liouville's Theorem. Every bounded entire function is constant.

Proof. If $|f(z)| \leq M$ for all $z \in \mathbb{C}$, apply the differentiated Cauchy formula on a circle of radius R:

$$|f'(z_0)| = \left| \frac{1}{2\pi i} \oint_{|z-z_0|=R} \frac{f(z)}{(z - z_0)^2}\, dz \right| \leq \frac{M}{R}$$

Letting $R \to \infty$ gives $f'(z_0) = 0$ for all z_0, so f is constant. $\square$

Corollary (Fundamental Theorem of Algebra). Every non-constant polynomial $p(z) \in \mathbb{C}[z]$ has at least one root in $\mathbb{C}$.

Proof. If $p(z) \neq 0$ for all z, then $f(z) = 1/p(z)$ is entire. Since $|p(z)| \to \infty$ as $|z| \to \infty$, f is bounded. By Liouville, f is constant, contradicting the non-constancy of p. $\square$

Taylor and Laurent Series

Theorem. If f is holomorphic on a disk $D(z_0, R)$, then f has a convergent power series expansion on that disk:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

This Taylor series converges absolutely and uniformly on compact subsets of $D(z_0, R)$.

A function that equals its Taylor series on some open disk is called analytic at z_0. The profound theorem of complex analysis is:

$$\text{holomorphic} \iff \text{analytic}$$

In real analysis, C^∞ does not imply analytic (the function e^{-1/x^2} has all derivatives zero at $x = 0$ but is not identically zero). In complex analysis, this pathology cannot occur.

Laurent Series. When f has a singularity at z_0, a Taylor series is unavailable. Instead, on an annulus $0 < |z - z_0| < R$, f has a *Laurent series*:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \underbrace{\sum_{n=0}^{\infty} a_n (z - z_0)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}}_{\text{principal part}}$$

where:

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

The principal part contains the negative-power terms. Its structure at z_0 classifies the singularity.

Singularities and the Residue Theorem

A point z_0 is an *isolated singularity* of f if f is holomorphic in some punctured disk $0 < |z - z_0| < r$ but not at z_0 itself. The Laurent series classifies isolated singularities into three types:

Removable singularity. The principal part is absent (all $a_{-n} = 0$). The singularity is an artifact of the definition; f can be redefined at z_0 to make it holomorphic. Example: $\dfrac{\sin z}{z}$ at $z = 0$.

Pole of order m. The principal part is finite, with $a_{-m} \neq 0$ and $a_{-n} = 0$ for $n > m$:

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + \cdots$$

A pole of order 1 is called a simple pole. At a pole, $|f(z)| \to \infty$ as $z \to z_0$. Functions holomorphic except for isolated poles are called *meromorphic*.

Essential singularity. The principal part is infinite (infinitely many non-zero a_{-n}). Near an essential singularity, the behavior is wildly irregular: by the *Weierstrass–Casorati theorem*, the image of any punctured neighborhood of z_0 is dense in $\mathbb{C}$. The stronger *Picard's Great Theorem* asserts the image omits at most one complex value. Example: $e^{1/z}$ at $z = 0$.

The *residue* of f at an isolated singularity z_0 is the coefficient a_{-1} of $(z - z_0)^{-1}$ in the Laurent expansion:

$$\operatorname{Res}(f, z_0) = a_{-1} = \frac{1}{2\pi i} \oint_\gamma f(z)\, dz$$

for γ a small positively oriented circle around z_0.

Computational formulas:

- Simple pole: $\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$
- Pole of order m: $\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right]$
- If $f = p/q$ with $p(z_0) \neq 0$, $q(z_0) = 0$, $q'(z_0) \neq 0$: $\operatorname{Res}(f, z_0) = p(z_0)/q'(z_0)$

Theorem (Cauchy's Residue Theorem). Let f be meromorphic on and inside a positively oriented simple closed contour γ, with isolated singularities $z_1, \ldots, z_k$ inside γ. Then:

$$\oint_\gamma f(z)\, dz = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f, z_j)$$

This theorem unifies Cauchy's integral theorem (zero residues ⇒ zero integral) and Cauchy's integral formula (which computes residues of the form $f(z)/(z - z_0)^n$). It is the master tool of complex integration.

Conformal Mappings

A holomorphic function $f : U \to \mathbb{C}$ is conformal at z_0 if $f'(z_0) \neq 0$. Such a mapping preserves the angle and orientation between any two smooth curves intersecting at z_0. This is seen geometrically: the derivative $f'(z_0)$ acts on tangent vectors as multiplication by the complex number $f'(z_0) = |f'(z_0)| e^{i \arg f'(z_0)}$, which rotates by $\arg f'(z_0)$ and scales by $|f'(z_0)|$ — a conformal (angle-preserving) linear map. The Möbius transformation (or linear fractional transformation) is:

$$T(z) = \frac{az + b}{cz + d}, \qquad a, b, c, d \in \mathbb{C},\ ad - bc \neq 0$$

Möbius transformations are the automorphisms of the Riemann sphere $\hat{\mathbb{C}}$. They map circles and lines to circles and lines (lines being "circles through ∞"), preserve cross-ratios, and form a group under composition isomorphic to $\mathrm{PSL}(2, \mathbb{C})$.

Three special subclasses are translations ($z \mapsto z + b$), rotations and scalings ($z \mapsto az$), and inversions ($z \mapsto 1/z$). Every Möbius transformation is a composition of these.

Theorem (Riemann Mapping Theorem). Let $U \subsetneq \mathbb{C}$ be a simply connected domain with $U \neq \mathbb{C}$. Then there exists a bijective holomorphic function $f : U \to D(0,1)$ (the open unit disk), unique up to a Möbius transformation of the disk.

This theorem asserts that all simply connected proper subdomains of $\mathbb{C}$ are conformally equivalent to the unit disk. Its proof uses the normal families technique (*Montel's theorem*): the desired map is obtained as the extremum of a functional on a compact family of holomorphic functions. The Riemann Mapping Theorem is foundational for applications: it reduces problems on complicated domains to problems on the standard disk or half-plane.