Complex Numbers
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Complex analysis — also called the theory of functions of a complex variable — is the branch of mathematics that extends the ideas of real analysis (limits, differentiation, integration, series) to functions whose domain and codomain lie in the complex plane $\mathbb{C}$. Far from being a mere generalization, complex analysis is qualitatively richer than its real counterpart: a function that is once differentiable in the complex sense is automatically infinitely differentiable, expressible as a convergent power series, and constrained by global geometric properties that have no real analogue.

The Complex Number System

The complex numbers $\mathbb{C}$ are constructed as the set $\mathbb{R}^2$ equipped with the operations:

$$(a, b) + (c, d) = (a + c,\ b + d)$$

$$(a, b) \cdot (c, d) = (ac - bd,\ ad + bc)$$

Writing $(a, b) = a + bi$ where $i = (0, 1)$, the multiplication rule encodes $i^2 = -1$. The pair (a, b) is written in standard form $z = a + bi$, where:

- $a = \mathrm{Re}(z)$ is the real part,
- $b = \mathrm{Im}(z)$ is the imaginary part.

$\mathbb{C}$ is a field (satisfying all field axioms) and contains $\mathbb{R}$ as the subfield $\{a + 0i : a \in \mathbb{R}\}$. Unlike $\mathbb{R}$, it admits no total ordering compatible with its field structure — $\mathbb{C}$ is not an ordered field.

The Complex Plane and Polar Form

Identifying $z = a + bi$ with the point $(a, b) \in \mathbb{R}^2$ yields the *Argand plane* (or *Gaussian plane*), with the real axis horizontal and the imaginary axis vertical.

The modulus (absolute value) of z is:

$$|z| = \sqrt{a^2 + b^2}$$

The argument of $z \neq 0$ is the angle $\theta = \arg(z)$ that the vector from the origin to z makes with the positive real axis. The polar form is:

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta), \qquad r = |z|,\ \theta = \arg(z)$$

where *Euler's formula* $e^{i\theta} = \cos\theta + i\sin\theta$ is justified by the agreement of power series:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} = \cos\theta + i\sin\theta$$

The principal argument $\mathrm{Arg}(z) \in (-\pi, \pi]$ is the unique argument in this range. The argument is multi-valued in general; incrementing θ by 2π yields the same point, a source of the subtleties surrounding multi-valued functions.

The *complex conjugate* of $z = a + bi$ is $\bar{z} = a - bi$. Key identities include:

$$z\bar{z} = |z|^2, \qquad \mathrm{Re}(z) = \frac{z + \bar{z}}{2}, \qquad \mathrm{Im}(z) = \frac{z - \bar{z}}{2i}$$

The multiplicative inverse of $z \neq 0$ is:

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

The *triangle inequality* holds: $|z + w| \leq |z| + |w|$.

By adding a single point at infinity ∞, one obtains the *Riemann sphere* (or extended complex plane) $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, topologically equivalent to the unit sphere S^2 via stereographic projection. This compactification allows poles and behavior at infinity to be treated uniformly — a meromorphic function on $\hat{\mathbb{C}}$ is simply a rational function.

Topology of the Complex Plane

Since $\mathbb{C} \cong \mathbb{R}^2$ as a metric space (with metric $d(z, w) = |z - w|$), the topological concepts of real analysis carry over directly.

- An open disk (or open ball) of radius r centered at z_0 is $D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.
- A set $U \subseteq \mathbb{C}$ is open if every point has an open disk entirely contained in U.
- A set is closed if its complement is open, and compact if it is closed and bounded.
- A domain (or region) is a non-empty, connected open set. Connectedness here means the set cannot be partitioned into two non-empty disjoint open subsets — equivalently, any two points can be joined by a path lying entirely in the set.

These topological concepts determine the appropriate domains for the major theorems of complex analysis.

Elementary Holomorphic Functions

The complex exponential is defined by the absolutely convergent power series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad z \in \mathbb{C}$$

It satisfies $e^{z+w} = e^z e^w$, is entire, and agrees with the real exponential on the real axis. Euler's formula gives $e^{x+iy} = e^x(\cos y + i \sin y)$, so $|e^{i\theta}| = 1$ for $\theta \in \mathbb{R}$.

The exponential is periodic with period $2\pi i$: $e^{z+2\pi i} = e^z$. This periodicity is the algebraic source of the multi-valuedness of logarithms and powers.

The complex trigonometric functions are defined via the exponential:

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

and the hyperbolic functions:

$$\cosh z = \frac{e^{z} + e^{-z}}{2}, \qquad \sinh z = \frac{e^{z} - e^{-z}}{2}$$

These satisfy the relation $\cos(iz) = \cosh z$, unifying the real trigonometric and hyperbolic functions under a single complex framework.

The complex logarithm is the multi-valued inverse of the exponential. For $z \neq 0$:

$$\log z = \ln|z| + i\arg(z)$$

Since $\arg(z)$ is defined only up to multiples of 2π, $\log z$ is multi-valued. The principal value is:

$$\operatorname{Log} z = \ln|z| + i\operatorname{Arg}(z), \qquad \operatorname{Arg}(z) \in (-\pi, \pi]$$

Log is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$ — the complex plane with the branch cut along the negative real axis removed. A branch cut is a curve along which a multi-valued function is rendered single-valued and continuous at the cost of a discontinuity across the cut.

For z, $\alpha \in \mathbb{C}$ with $z \neq 0$, define complex power as:

$$z^{\alpha} = e^{\alpha \log z}$$

This is again multi-valued unless $\alpha \in \mathbb{Z}$. The principal value uses Log. For example,

$$i^{i} = e^{i \operatorname{Log} i} = e^{i \cdot i\pi/2} = e^{-\pi/2} \in \mathbb{R}.$$

Theorem (Identity Theorem). If f and g are holomorphic on a connected open set U, and $f(z) = g(z)$ on a set with an accumulation point in U, then $f \equiv g$ on all of U.

An accumulation point of a set S is a point p such that every neighborhood of p contains a point of S other than p itself. The identity theorem encodes the rigidity of holomorphic functions: they are completely determined by their values on any convergent sequence. This has no real analogue — two real-smooth functions can agree on an interval and differ elsewhere.

Analytic continuation is the process of extending a holomorphic function beyond its original domain of definition, exploiting the identity theorem to ensure uniqueness. Given f holomorphic on a domain U, a direct analytic continuation to an overlapping domain V is a holomorphic function g on V such that $f = g$ on $U \cap V$.

The *monodromy theorem* governs when analytic continuation along different paths yields the same result: if U is simply connected, the continuation is path-independent and yields a single-valued function.

Multi-valued functions like $\sqrt{z}$ and $\log z$ arise precisely when the domain is not simply connected; continuation along a loop encircling a branch point (*e.g.*, $z = 0$) returns a different branch.

The *Riemann surface* of a multi-valued function is the natural domain on which it becomes single-valued — a branched covering of $\mathbb{C}$ constructed by gluing multiple copies ("sheets") of the complex plane along branch cuts.

The *Riemann zeta function*, initially defined by the Dirichlet series for $\mathrm{Re}(s) > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

extends by analytic continuation to a meromorphic function on all of $\mathbb{C}$, with a simple pole $ats = 1$. The *Riemann Hypothesis* — that all non-trivial zeros lie on the critical line $\mathrm{Re}(s) = 1/2$ — is one of the Millennium Prize Problems, and its consequences for the distribution of prime numbers illustrate how deeply complex analysis penetrates number theory.

Just as a polynomial is determined (up to a constant) by its roots, an entire function is determined by its zeros via the *Weierstrass factorization theorem*:

Theorem. Every entire function f with zeros $z_1, z_2, \ldots$ (repeated with multiplicity) can be expressed as:

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$$

where $g(z)$ is entire, m is the order of the zero at the origin, and

$$E_p(z) = (1 - z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right)$$

are elementary factors chosen to ensure convergence.

For the sine function, this gives Euler's product formula:

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

from which, comparing Taylor coefficients, one recovers $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

For entire functions of finite order ρ (meaning $|f(z)| = O(e^{|z|^\rho})$), *Hadamard's theorem* sharpens the Weierstrass factorization by relating the degree of g to ρ, providing a complete structural description.