<u>Fundamental Theorem of Calculus</u>
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<u>Differentiation</u>

The derivative of f at x_0 is defined as the limit of the difference quotient:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists. If it does, f is said to be differentiable at x_0. Geometrically, $f'(x_0)$ is the slope of the tangent line to the graph of f at x_0.

Proposition. Differentiability implies continuity. The converse is false: $f(x) = |x|$ is continuous at 0 but not differentiable there.

<u>Mean Value Theorem (MVT).</u> If f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The MVT asserts that the instantaneous rate of change at some interior point equals the average rate of change over the interval. It is the primary tool for relating function values to derivative information.

Corollaries of the MVT:

If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant on $[a, b]$.
If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on $[a, b]$.

<u>Taylor's Theorem</u>. For a function f that is $(n + 1)$-times differentiable on an interval containing a, Taylor's theorem provides a polynomial approximation with an explicit error term:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x - a)^k + R_n(x)$$

where the *Lagrange remainder* is:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}(x - a)^{n+1}$$

for some c between x and a. The partial sum $\sum_{k=0}^{n}$ is the Taylor polynomial of degree n. When $a = 0$, it is called a *Maclaurin polynomial*.

Taylor's theorem is the mathematical foundation for linearization, small-angle approximations in mechanics, and the derivation of finite-difference schemes in numerical analysis.

Riemann Integration

Let $f : [a, b] \to \mathbb{R}$ be bounded. A partition P of $[a, b]$ is a finite set of points $a = x_0 < x_1 < \cdots < x_n = b$. For each subinterval $[x_{i-1}, x_i]$, define:

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x), \qquad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$$

The upper Riemann sum and lower Riemann sum are:

$$U(f, P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}), \qquad L(f, P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1})$$

The upper integral and lower integral are:

$$\overline{\int_a^b} f = \inf_P U(f, P), \qquad \underline{\int_a^b} f = \sup_P L(f, P)$$

Riemann Integrability. f is Riemann integrable on $[a, b]$ if the upper and lower integrals coincide, in which case:

$$\int_a^b f(x)\, dx = \overline{\int_a^b} f = \underline{\int_a^b} f$$

Riemann's Criterion for Integrability. f is Riemann integrable on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that

$$U(f, P) - L(f, P) < \varepsilon$$

Key integrability results:
Every continuous function on $[a, b]$ is Riemann integrable.
Every monotone bounded function on $[a, b]$ is Riemann integrable.
A bounded function is Riemann integrable if and only if its set of discontinuities has measure zero (Lebesgue's characterization).

The Fundamental Theorem of Calculus (FTC)

This theorem unifies differentiation and integration — the two pillars of calculus.

FTC Part I (Antiderivative). If f is Riemann integrable on $[a, b]$ and $F(x) = \displaystyle\int_a^x f(t)\, dt$, then F is continuous on $[a, b]$. Moreover, if f is continuous at $x_0 \in (a, b)$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

FTC Part II (Evaluation). If f is Riemann integrable on $[a, b]$ and G is any antiderivative of f (*i.e.*, inverse $G' = f$) on $[a, b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

Metric Spaces

A metric space (X, d) is a set X together with a metric $d : X \times X \to \mathbb{R}$ satisfying for all $x, y, z \in X$:

1. Non-negativity: $d(x, y) \geq 0$, with $d(x, y) = 0 \iff x = y$
2. Symmetry: $d(x, y) = d(y, x)$
3. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$

The real line $\mathbb{R}$ with $d(x, y) = |x - y|$ is the prototypical example, but the framework extends to function spaces, sequence spaces, and more.

A metric space is complete if every Cauchy sequence converges. $\mathbb{R}$ is complete; $\mathbb{Q}$ is not.

A subset K of a metric space is compact if every open cover of K has a finite subcover. In $\mathbb{R}^n$, by the *Heine–Borel theorem*, compactness is equivalent to being closed and bounded.

Compact sets are the natural domains for the strongest theorems about continuous functions (Extreme Value Theorem, uniform continuity).

Lebesgue Integration

While Riemann integration is adequate for continuous functions and many engineering applications, it fails for important limiting processes. The *Lebesgue integral*, developed by Henri Lebesgue (1902), extends integration to a vastly larger class of functions.

Measure Theory. A σ-algebra on a set X is a collection $\mathcal{F}$ of subsets of X closed under complement and countable unions. A measure $\mu : \mathcal{F} \to [0, +\infty]$ satisfies:

- $\mu(\emptyset) = 0$
- Countable additivity: If $\{A_n\}$ are pairwise disjoint, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

The Lebesgue measure λ on $\mathbb{R}$ assigns $\lambda([a, b]) = b - a$, consistent with ordinary length.

The Lebesgue Integral. For a measurable function $f \geq 0$, the Lebesgue integral is constructed by approximating f from below by simple functions (finite linear combinations of indicator functions):

$$\int f\,d\mu = \sup\left\{\int \phi\,d\mu : 0 \leq \phi \leq f,\ \phi \text{ simple}\right\}$$

The real power of the Lebesgue integral lies in its exchange-of-limits theorems.

Monotone Convergence Theorem. If $0 \le f_1 \le f_2 \le \cdots$ and $f_n \to f$ pointwise, then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

Dominated Convergence Theorem (DCT). If $f_n \to f$ pointwise and $|f_n(x)| \le g(x)$ for an integrable dominating function g (*i.e.*, $\displaystyle\int g \, d\mu < \infty$), then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$$

The DCT is one of the most applied theorems in analysis, justifying differentiation under the integral sign and validating many operations in Fourier analysis and probability theory.

Applications in Science and Engineering

Fourier Analysis and Signal Processing: The Fourier series of a 2π-periodic function f is:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

where the Fourier coefficients are:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx$$

The convergence of these series — pointwise, uniform, or in the L^2 sense — is a central question of real analysis. The L^2 theory (using the Lebesgue integral) gives a complete orthonormal basis $\{e^{inx}/\sqrt{2\pi}\}$ for the Hilbert space of square-integrable functions, foundational to signal processing and quantum mechanics.

Parseval's Theorem states:

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{|a_0|^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

expressing the conservation of energy between time and frequency domains.

Ordinary Differential Equations: The existence and uniqueness of solutions to initial value problems

$$\frac{dy}{dt} = F(t, y), \qquad y(t_0) = y_0$$

is guaranteed by the *Picard–Lindelöf theorem*, which requires F to be continuous in t and Lipschitz continuous in y: $\exists\, K > 0$ such that

$$|F(t, y_1) - F(t, y_2)| \leq K|y_1 - y_2|$$

The proof uses the *Banach Fixed Point Theorem* (or Contraction Mapping Theorem) in the complete metric space of continuous functions — a direct application of completeness.

Numerical Analysis and Error Bounds: Taylor's theorem quantifies the error in polynomial approximations. For a function approximated by its degree-n Taylor polynomial T_n:

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}, \qquad M = \max_{c\in[a,x]} |f^{(n+1)}(c)|$$

This bound is used to determine how many terms are needed to achieve a prescribed accuracy — *e.g.*, in computing $\sin, \cos, e^x$ in embedded systems.

The Newton–Raphson method for solving $f(x) = 0$ iterates

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Its quadratic convergence (the error roughly squares at each step) is proved using the MVT and Taylor's theorem.

Probability and Statistics: The foundation of modern probability theory, formalized by Kolmogorov (1933), is measure theory. A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a measure space with $\mathbb{P}(\Omega) = 1$. The expectation of a random variable X is the Lebesgue integral:

$$\mathbb{E}[X] = \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$$

The *Law of Large Numbers* and the *Central Limit Theorem* — which asserts that the normalized sample mean of *i.i.d.* random variables with mean μ and variance σ^2 converges in distribution to $\mathcal{N}(0,1)$:

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0,1)$$

— are both proved using the DCT and characteristic functions (Fourier transforms of probability measures).

Partial Differential Equations: PDEs such as the heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$$

and the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

are solved via separation of variables and *Fourier* series. The well-posedness (existence, uniqueness, and stability of solutions) relies on the L^2 theory and Sobolev spaces — function spaces defined using weak derivatives and Lebesgue norms of the form $\|f\|_{W^{k,p}} = \left(\sum_{|\alpha| \le k} \int |D^\alpha f|^p \right)^{1/p}$.

Summary of Logical Dependencies

The logical architecture of real analysis forms a strict hierarchy:

$$\begin{aligned}
&\text{Completeness Axiom} \\
&\Longrightarrow \text{Cauchy criterion, BWT, MCT} \\
&\Longrightarrow \text{Continuity theorems (EVT, IVT)} \\
&\Longrightarrow \text{Differentiability (MVT, Taylor)} \\
&\Longrightarrow \text{Riemann integral, FTC} \\
&\Longrightarrow \text{Uniform convergence, power series} \\
&\xrightarrow{\text{extend}} \text{Lebesgue measure, } L^p \text{ spaces} \\
&\Longrightarrow \text{Fourier analysis, probability, PDEs}
\end{aligned}$$

Every theorem in this chain is a consequence of the axioms of $\mathbb{R}$, chiefly the completeness axiom, which is what makes the real numbers the correct arena for analysis. Remove completeness, and the edifice collapses: limits fail to exist, integrals become ill-defined, and differential equations lose unique solutions.

Concluding Remarks

Real analysis is not merely a catalog of theorems; it is a discipline of justified reasoning. The $\varepsilon - \delta$ formalism trains the practitioner to ask: under precisely what conditions does this limiting process behave as expected? The answer invariably traces back to completeness, compactness, or some form of uniform control — concepts that, once internalized, illuminate the reliability (and the limits) of every analytical tool used in science and engineering.